A forbidden induced subgraph characterization of distance-hereditary 5-leaf powers

Andreas Brandstädt\textsuperscript{a}, Van Bang Le\textsuperscript{a}, Dieter Rautenbach\textsuperscript{b,∗}

\textsuperscript{a} Institut für Informatik, Universität Rostock, D-18051 Rostock, Germany
\textsuperscript{b} Institut für Mathematik, TU Ilmenau, D-98684 Ilmenau, Germany

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\textbf{A B S T R A C T}

A graph $G$ is a $k$-\textit{leaf power} if there is a tree $T$ such that the vertices of $G$ are the leaves of $T$ and two vertices are adjacent in $G$ if and only if their distance in $T$ is at most $k$. In this situation $T$ is called a $k$-\textit{leaf root} of $G$. Motivated by the search for underlying phylogenetic trees, the notion of a $k$-leaf power was introduced and studied by Nishimura, Ragde and Thilikos and subsequently in various other papers. While the structure of 3- and 4-leaf powers is well understood, for $k \geq 5$ the characterization of $k$-leaf powers remains a challenging open problem.

In the present paper, we give a forbidden induced subgraph characterization of distance-hereditary 5-leaf powers. Our result generalizes known characterization results on 3-leaf powers since these are distance-hereditary 5-leaf powers.

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\section{1. Introduction}

The reconstruction of the evolutionary history of a set of species, based on quantitative biological data, is one of the challenging problems in computational biology. Typically, the evolutionary history is modeled by an evolutionary tree called \textit{phylogeny} which is a tree whose leaves are labeled by species and each internal node represents a speciation event whereby an ancestral species gives rise to two or more child species [2,6,13].

Motivated by this background, Nishimura, Ragde and Thilikos [11] introduced the notion of a $k$-\textit{leaf power} and a $k$-\textit{leaf root} which are defined for a finite undirected graph $G = (V_G, E_G)$ and an integer $k \geq 2$ as follows: $G$ is a $k$-\textit{leaf power} if there exists a tree $T$ with $V_G$ as its set of leaves such that for all $x, y \in V_G$ with $x \neq y$ we have $xy \in E_G$ if and only if the distance of $x$ and $y$ in $T$ is at most $k$. In this situation $T$ is called a $k$-\textit{leaf root} of $G$.

Since leaves at distance two in a $k$-leaf root $T$ of some graph $G$ obviously play equivalent roles within $G$, it often simplifies the statements and arguments considerably to consider so-called \textit{basic $k$-leaf roots} which are $k$-leaf roots in which no two leaves are at distance two. If a graph $G$ has a basic $k$-leaf root, then it is called a \textit{basic $k$-leaf power}. Note that every $k$-leaf power is obtained from a basic $k$-leaf power $G$ by replacing the vertices of $G$ by cliques.

Obviously, a graph is a 2-leaf power if and only if it is the disjoint union of cliques or – equivalently – it is $P_3$-free. In [7], 3-leaf powers were characterized in terms of forbidden induced subgraphs and in [3,12], it was shown that 3-leaf powers are exactly those graphs which are obtained from a tree by replacing its vertices by cliques. In [5,12], characterizations of 4-leaf powers were given. In [11], very complicated $\Theta(n^3)$ time algorithms for recognizing 3-leaf powers and 4-leaf powers, respectively, and constructing 3-leaf roots and 4-leaf roots, respectively, if they exist, were described. Based on structural...
results, simple linear time algorithms for these problems were described in [3,5]. For \( k \geq 3 \), no characterization of \( k \)-leaf powers is known despite considerable effort. Even the characterization of 5-leaf powers appears to be a major open problem.

In the present paper, we characterize basic distance-hereditary 5-leaf powers in terms of their block structure and gluing conditions. It turns out that the blocks are either 3-leaf powers or have a structure based on the dart or the bull as induced subgraph (cf. Fig. 1). The structure of the blocks as well as the gluing conditions can be expressed in terms of the 34 forbidden induced subgraphs \( F_1, \ldots, F_{34} \) shown in Figs. 7 and 9–11. In particular, two of our main results are the following:

(i) \( G \) is a distance-hereditary 2-connected basic 5-leaf power if and only if \( G \) is chordal and contains no induced \( F_1, \ldots, F_8 \).

(ii) \( G \) is a distance-hereditary basic 5-leaf power if and only if \( G \) is chordal and contains no induced \( F_1, \ldots, F_{34} \).

Furthermore, we explain how the set of all minimal forbidden induced subgraphs different from the chordless cycles for the – not necessarily basic – distance-hereditary 5-leaf powers can easily be generated from \( F_1, \ldots, F_{34} \). Since this list is very long and does not yield further structural insight, we will not give it explicitly. Our results extend the known characterization of 3-leaf powers [7,3] which are distance-hereditary basic 5-leaf powers and is a non-trivial step towards a characterization of 5-leaf powers in general. We hope that our approach can be extended to such a characterization.

2. Notation

Throughout this paper, we consider finite undirected graphs \( G = (V_G, E_G) \) without loops or multiple edges, with vertex set \( V_G \) and edge set \( E_G \). For a vertex \( v \in V_G \), let \( N_G(v) = \{u \mid uv \in E_G\} \) denote the neighborhood of \( v \) in \( G \), and let \( N_G[v] = \{v\} \cup N_G(v) \) denote the closed neighborhood of \( v \) in \( G \). The degree of a vertex \( v \) is the number of its neighbors, i.e., \( |N_G(v)| \). A clique is a set of mutually adjacent vertices. A stable set is a set of mutually non-adjacent vertices. The complement of \( G \) is denoted by \( \overline{G} \).

The maximal induced 2-connected subgraphs of \( G \) are the blocks of \( G \). A vertex whose removal increases the number of components is a cutvertex. A block of \( G \) which contains at most one cutvertex is an endblock.

For \( U \subseteq V \), let \( G[U] \) denote the subgraph of \( G \) induced by \( U \). If \( \mathcal{F} \) denotes a set of graphs, then a graph \( G \) is \( \mathcal{F} \)-free if none of its induced subgraphs is in \( \mathcal{F} \).

Two vertices \( x, y \in V \) are true twins if \( N_G[x] = N_G[y] \). A vertex set \( U \subseteq V_G \) is a module of \( G \) if \( U \subseteq N_G(v) \) or \( U \cap N_G(v) = \emptyset \) for all \( v \in V_G \setminus U \). A homogeneous set of \( G \) is a module which consists of at least two, but not all vertices of \( G \). A clique module in \( G \) is a module which is a clique in \( G \). Obviously, true twins form a clique module. In [10], a critical clique is defined as a maximal clique module and the critical clique graph of \( G \), denoted by \( cc(G) \), is defined as the graph whose vertices are the critical cliques of \( G \) and two vertices of \( cc(G) \) are adjacent whenever the two corresponding critical cliques of \( G \) contain adjacent vertices. (Note that a critical clique graph cannot contain true twins while a basic \( k \)-leaf power can.)

Replacing a vertex \( v \) in a graph \( G \) by a graph \( H \) results in the graph obtained from \( G[V_G \setminus \{v\}] \cup H \) by adding all edges between vertices in \( N_G(v) \) and vertices in \( V_H \).

For a positive integer \( k \geq 1 \), let \( K_k \) denote the complete graph with \( k \) vertices, let \( P_k \) denote the chordless path with \( k \) vertices and \( k - 1 \) edges, and for \( k \geq 3 \), let \( C_k \) denote the chordless cycle with \( k \) vertices and \( k \) edges. A graph is chordal if it contains no induced \( C_k \) with \( k \geq 4 \).

Let \( dc(x, y) \) denote the minimum number of edges of a path in \( G \) between \( x \) and \( y \), i.e., the distance of \( x \) and \( y \) in \( G \). A graph \( G \) is distance-hereditary if the distance between two vertices in every connected induced subgraph \( H \) of \( G \) is the same in \( H \) and in \( G \).

In [8] it was shown that a graph is distance-hereditary if and only if each cycle of length at least five has two crossing chords which implies that a chordal graph is distance-hereditary if and only if it is gem-free (cf. Fig. 1). A graph is a block graph if each of its blocks is a clique. Clearly, a chordal graph is a block graph if and only if it is diamond-free (cf. Fig. 1). See [4] for more details on terminology and these graph classes.

It is known that \( k \)-leaf powers are chordal [3,7,12] (indeed, they are ‘strongly’ chordal; cf. [5]—for a kind of converse cf. [9]), and 3-leaf powers have been characterized as follows.

**Theorem 1** ([3,7,12]). For a connected graph \( G \) the following statements are equivalent.

(i) \( G \) is a 3-leaf power;
(ii) \( G \) arises from a tree by replacing the vertices by cliques;
(iii) \( cc(G) \) is a tree;
(iv) \( G \) is \( \langle \text{dart, bull, gem} \rangle \)-free chordal.

The following observations are obvious and imply that the class of (basic) \( k \)-leaf powers can be characterized by forbidden induced subgraphs, and that we can restrict to connected graphs when considering \( k \)-leaf powers.
Fig. 2. Extended dart (left) and its basic 5-leaf roots.

Fig. 3. Extended bull (left) and its basic 5-leaf root.

Observation 1. (i) Every induced subgraph of a basic k-leaf power is a basic k-leaf power.
(ii) A k-leaf power without true twins is a basic k-leaf power.
(iii) A graph is a k-leaf power if and only if each of its connected components is a k-leaf power.

3. Preparatory results

The two types of graphs whose structure is based on the dart and the bull which will play a central role for our investigation are the following (cf. Figs. 2, 3, 5 and 6).

Definition 1. (i) A plump dart is a graph arising from the dart by replacing each of the vertices of degree 1 or degree 2 by a non-empty union of cliques, the vertex of degree 3 by a non-empty clique, and the vertex of degree 4 by a $K_2$.

Vertices of a plump dart that replaced the vertices of degree 3 or degree 4 of the original dart are called inner vertices of the plump dart.

The plump dart of minimum order is called the extended dart; cf. Fig. 2(left).

(ii) A plump bull is a graph arising from the bull by replacing each of the two cutvertices by a $K_2$, the vertices of degree 1 by a non-empty union of cliques, and the vertex of degree 2 by a non-empty clique.

Vertices of a plump bull that replaced the vertices of degree 2 or degree 3 of the original bull are called inner vertices of the plump bull.

The plump bull of minimum order is called the extended bull; cf. Fig. 3(left).

The plump darts and the plump bulls have quite well-behaved basic 5-leaf roots whose properties are analysed in the following lemma.

Lemma 1. (i) The extended dart has exactly the basic 5-leaf roots depicted in Fig. 2. (Note that there is a certain degree of freedom which leaves of the root correspond to which vertices of the extended dart.)
(ii) The extended bull has the unique basic 5-leaf root depicted in Fig. 3.
(iii) Plump darts (plump bulls) are basic 5-leaf powers and for every basic 5-leaf root $T$ and every inner vertex $u$ of a plump dart (plump bull) there exists another vertex $v$ of the plump dart (plump bull) with $d_T(u, v) = 3$.

Proof. (i): Let $G$ be the extended dart and let $T$ be a basic 5-leaf root of $G$. We denote the vertices of $G$ as shown in Fig. 2.

Since $v_2v_6 \notin E_G$, we have $d_T(v_2, v_6) \geq 6$. If $d_T(v_2, v_6) \geq 8$, then $T$ can have at most one leaf which is at distance at most 5 from $v_2$ as well as $v_6$ which contradicts $|N_C(v_2) \cap N_C(v_6)| \geq 2$. Hence $d_T(v_2, v_6) \leq 7$.

If $\max\{d_T(v_2, v_4), d_T(v_2, v_5)\} = 5$, then $d_T(v_1, v_2), d_T(v_1, v_4), d_T(v_1, v_5) \leq 5$ and $d_T(v_3, v_2), d_T(v_3, v_4), d_T(v_3, v_5) \leq 5$ imply $d_T(v_1, v_3) \leq 5$ which contradicts $v_1v_3 \notin E_G$. Hence $d_T(v_2, v_4), d_T(v_2, v_5) \leq 4$ which implies $d_T(v_2, v_6) = 6, \{d_T(v_2, v_4), d_T(v_2, v_5)\} = \{3, 4\}$ and $\{d_T(v_6, v_4), d_T(v_6, v_5)\} = \{4, 5\}$. Note that these conditions uniquely determine the subtree of $T$ with leaves $v_2, v_4, v_5$ and $v_6$ up to exchanging the positions of $v_4$ and $v_5$.

Since $v_2, v_4, v_5 \in N_C(v_1)$ but $v_6 \notin N_C(v_1)$, we have that either $d_T(v_1, v_2) = 5$ and $\{d_T(v_1, v_4), d_T(v_1, v_5)\} = \{4, 5\}$ or $d_T(v_1, v_2) = 3$ and $\{d_T(v_1, v_4), d_T(v_1, v_5)\} = \{4, 5\}$. By symmetry, the same conditions hold for $v_3$. Note again that these conditions determine the possible positions of $v_1$ and $v_3$ within $T$ and hence also their possible mutual distances. Since $d_T(v_1, v_3) \geq 6$, we obtain that either $d_T(v_1, v_2) = d_T(v_3, v_2) = 5$ or $\{d_T(v_1, v_2), d_T(v_3, v_2)\} = \{3, 5\}$ which uniquely determines the two possibilities for $T$ shown in Fig. 2 up to exchanging the positions of $v_4$ and $v_5$ and the positions of $v_1$ and $v_3$.

(ii): Let $G$ be the extended bull and let $T$ be a basic 5-leaf root of $G$. We denote the vertices of $G$ as shown in Fig. 3.
Lemma 1 easily implies that the graphs in Fig. 7 are forbidden induced subgraphs for basic 5-leaf powers.

**Lemma 2.** The graphs $F_1, \ldots, F_8$ in Fig. 7 are not basic 5-leaf powers.

**Proof.** Note that the graphs $F_1, \ldots, F_5$ contain an extended dart as an induced subgraph and that the graphs $F_4, \ldots, F_8$ contain an extended bull as an induced subgraph.

The proof now follows easily by considering all basic 5-leaf roots of the extended dart and the extended bull as described by Lemma 1. We will give the details only for the graphs $F_1$ and $F_4$ and leave the very simple and similar rest to the reader.

Let $T$ be a basic 5-leaf root of $F_1$. We denote the vertices of the induced extended dart in $F_1$ as in Fig. 2 and the additional vertex by $v_7$. By Lemma 1, we have $d_T(v_1, v_6) = 7$. Since $v_1, v_6 \in \mathcal{N}_T(v_7)$, this implies (by the two possibilities described in Lemma 1(i)) that $\min\{d_T(v_4, v_7), d_T(v_5, v_7)\} = 2$ which is a contradiction.

The graph $F_4$ arises from the extended bull (cf. Fig. 3) by adding a new vertex $v_8$ adjacent to $v_2, v_3, v_4$ and $v_5$, i.e., $\{v_2, v_3, v_4, v_5, v_8\}$ forms a clique in $F_4$. Since any potential basic 5-leaf root of $F_4$ necessarily contains the basic 5-leaf
root of the extended bull, and since \( v_8 \) is non-adjacent to \( v_7 \), the basic 5-leaf root of the extended bull depicted in Fig. 3 (right) shows that \( F_4 \) cannot be a basic 5-leaf power and the proof is complete. \( \Box \)

We close this section with a useful observation about 2-connected distance-hereditary graphs.

**Observation 2.** If \( G \) is a 2-connected distance-hereditary graph, then \( |N_G(v) \cap N_G(u)| \neq 1 \) for all vertices \( u, v \in V_G, u \neq v \) with \( uv \notin E_G \).

**Proof.** We assume to the contrary that there are vertices \( u, v \in V_G, u \neq v \) with \( uv \notin E_G \) such that \( N_G(v) \cap N_G(u) = \{w\} \). Since \( G \) is 2-connected, there is a shortest path \( P \) from \( v \) to \( u \) avoiding \( w \). Since \( w \) is the only neighbor of \( v \) in \( N_G(u) \), the length of \( P \) is at least 3 which clearly contradicts the assumption that \( G \) is distance-hereditary. \( \Box \)

4. The blocks of distance-hereditary basic 5-leaf powers

In this section we characterize the blocks of distance-hereditary basic 5-leaf powers. Our argument relies on the analysis of the neighborhood of certain homogeneous sets.

**Definition 2.** Let \( G \) be a graph, \( A \subseteq V_G \) be a homogeneous set in \( G \) and let \( C = N_G(A) \setminus A \). The homogeneous set \( A \) is called a special homogeneous set of \( G \), if \( A \) is non-complete and there is a vertex \( u \in V_G \setminus (A \cup C) \) such that \( \emptyset \neq N_G(u) \cap C \neq C \).

**Lemma 3.** Let \( G \) be a 2-connected distance-hereditary \( \{F_1, F_2, F_3, F_4, F_5\} \)-free chordal graph. If \( G \) has a special homogeneous set, then \( G \) is a plump dart.

**Proof.** Let \( A_0 \subseteq V_G \) be an inclusion-minimal special homogeneous set in \( G \). Let

\[
C = N_G(A_0) \setminus A_0, A_1 = \{u \in V_G \setminus (A_0 \cup C) \mid N_G(u) \cap C = C\},
\]

and

\[
B = \{u \in V_G \setminus (A_0 \cup C) \mid \emptyset \neq N_G(u) \cap C \neq C\}.
\]

Since \( G \) is chordal and \( A_0 \) is non-complete, \( C \) is complete.

Considering a vertex in \( A_0 \), Observation 2 implies that every vertex in \( B \) has at least two neighbors in \( C \). Hence, by the definition of \( B \), we obtain \( |C| \geq 3 \).

If \( G[A_0] \) is not the disjoint union of cliques, then let \( a_1a_2a_3 \) be an induced \( P_3 \) in \( A_0 \). If \( a_2 \) is adjacent to all vertices in \( A_0 \setminus \{a_2\} \), then \( A_0 \setminus \{a_2\} \) is a special homogeneous set, which contradicts the choice of \( A_0 \). Hence there is a vertex \( a_4 \in A_0 \setminus \{a_2\} \) that is non-adjacent to \( a_2 \). If \( a_4 \) is non-adjacent to \( a_1 \) and \( a_3 \), then \( a_1, a_2, a_3, a_4 \) and three vertices of \( C \) induce the graph \( F_1 \), which is a contradiction. Hence \( a_4 \) is adjacent either to \( a_1 \) or to \( a_3 \). If \( a_4 \) is adjacent to \( a_1 \), but not to \( a_3 \), then \( a_1, a_2, a_3, a_4 \) and a vertex from \( C \) induce a gem, which is a contradiction. Hence \( a_4 \) is adjacent to both, \( a_1 \) and \( a_3 \), and \( a_1, a_2, a_3, a_4 \) induce a chordless cycle of length four, which is a contradiction. Altogether, it follows that \( G[A_0] \) is the disjoint union of cliques. Since \( A_0 \) is special, there are two distinct non-adjacent vertices \( a' \) and \( a'' \) in \( A_0 \).

Since \( G \) does not contain \( F_4 \) as an induced subgraph, every vertex in \( B \) has exactly two neighbors in \( C \). If two vertices \( b_1 \) and \( b_2 \) in \( B \) have disjoint neighborhoods in \( C \), we obtain a contradiction either to the chordality of \( G \) (\( b_1b_2 \in E_G \)) or to the assumption that \( G \) does not contain \( F_4 \) as an induced subgraph (\( b_1b_2 \notin E_G \)). Hence every two vertices in \( B \) have a common neighbor in \( C \). If \( b_1 \) and \( b_2 \) in \( B \) are such that \( N_G(b_1) \cap C = \{c_0, c_1\} \) and \( N_G(b_2) \cap C = \{c_0, c_2\} \) with \( c_1 \neq c_2 \), then \( G[c_0, c_1, c_2, b_1, b_2] \) is a gem or \( G[c_1, c_2, b_1, b_2] \) is a \( C_4 \), which is a contradiction. Hence there are two distinct vertices \( c' \) and \( c'' \) in \( C \) such that \( N_G(u) \cap C = \{c', c''\} \) for all \( u \in B \).

If \( G[A_1] \) is not the disjoint union of cliques, then three vertices from \( A_1 \), that induce a \( P_3 \) together with three vertices from \( C \) and the vertex \( a' \) induce the graph \( F_1 \), which is a contradiction. Hence \( G[A_1] \) is the disjoint union of cliques.
If \( G[B] \) is not the disjoint union of cliques, then three vertices from \( B \) that induce a \( P_3 \) together with \( c', c'' \), a third vertex from \( C, a' \), and \( a'' \) induce the graph \( F_2 \), which is a contradiction. Hence \( G[B] \) is the disjoint union of cliques.

If there is an edge \( ab \) with \( a \in A_1 \) and \( b \in B \), then \( a \) and \( b \) together with \( c' \), a vertex from \( C \) different from \( c'' \) and the vertex \( a' \) induce a gem, which is a contradiction. Hence there are no edges between the sets \( A_1 \) and \( B \).

By Observation 2 applied to the vertex \( c' \), no vertex in \( V_G \setminus (A_0 \cup C \cup A_1 \cup B) \) has exactly one neighbor in \( A_1 \cup B \). If there is a vertex \( u \in V_G \setminus (A_0 \cup C \cup A_1 \cup B) \) that has two neighbors \( a_1, a_2 \) in \( A_1 \), then \( a_1 a_2 \in E_G \) and \( u, a_1, a_2 \), a vertex from \( B, c', c'' \), a third vertex from \( C, \) and the vertex \( a' \) induce the graph \( F_3 \), which is a contradiction. If there is a vertex \( u \in V_G \setminus (A_0 \cup C \cup A_1 \cup B) \) that has two neighbors \( b_1, b_2 \) in \( B \), then \( b_1 b_2 \in E_G \) and \( u, b_1, b_2, c', c'' \), a third vertex from \( C, a' \) and \( a'' \) induce the graph \( F_3 \), which is a contradiction. Since \( G \) is chordal, no vertex in \( V_G \setminus (A_0 \cup C \cup A_1 \cup B) \) has a neighbor in \( A_1 \) and a neighbor in \( B \).

 Altogether, no vertex in \( V_G \setminus (A_0 \cup C \cup A_1 \cup B) \) has a neighbor in \( A_1 \cup B \) which implies that \( V_G = A_0 \cup C \cup A_1 \cup B \). Now, by contracting \( A_0, A_1, B, C \setminus \{ c', c'' \}, \) and \( \{ c', c'' \} \) to single vertices, we can conclude that \( G \) is a plump dart, and the proof is complete. \( \square \)

**Lemma 4.** Let \( G \) be a 2-connected distance-hereditary \( F_2 \)-free chordal graph. If \( G \) has no special homogeneous set, then \( cc(G) \) is a block graph.

**Proof.** For contradiction, we assume that \( cc(G) \) is not a block graph. Since \( cc(G) \) is connected and chordal, \( cc(G) \) contains a diamond \( D \) as an induced subgraph, say, with vertices \( v_1, v_2, v_3 \) and \( v_4 \), and edges \( v_1, v_2, v_3, v_1 v_4, v_2 v_3, \) and \( v_3 v_4 \). Since \( cc(G) \) is isomorphic to an induced subgraph of \( G \) by definition, we may assume for convenience that \( D \) is also an induced subgraph in \( G \).

Since \( v_1 \) and \( v_3 \) are no true twins (otherwise \( v_1 \) and \( v_3 \) would belong to a critical clique in \( G \) and hence would be contracted in \( cc(G) \)), we may assume that there is a vertex \( v_5 \in N_C(v_3) \setminus N_C(v_1) \). Since \( G \) is chordal and gem-free, \( v_5 \) is not adjacent to \( v_2 \) or to \( v_4 \).

Let \( A \) be the vertex set of the connected component of \( \overline{G}[N_C(v_1) \cap N_C(v_2)] \) that contains \( v_2 \) and \( v_4 \). Since \( A \) is a special homogeneous set in \( \overline{G}[[v_1, v_3, v_5] \cup (N_C(v_1) \cap N_C(v_2))] \) but not in \( G \), there is a vertex \( v_6 \notin A \) that is adjacent to some but not to all vertices of \( A \). By the definition of \( A, v_6 \notin N_C(v_1) \cap N_C(v_2) \). If \( v \) and \( w \) are two non-adjacent vertices in \( A \), then \( v_5 \) is adjacent neither to \( v \) nor to \( w \) because \( G \) is chordal and gem-free. Hence, by the connectedness of \( \overline{G}[A] \), we may assume without loss of generality, that \( v_6 \) is adjacent to \( v_2 \) but not to \( v_4 \). Since \( G \) is gem-free and \( v_6 \notin N_C(v_1) \cap N_C(v_2), v_6 \) is not adjacent to \( v_1 \) or to \( v_5 \). Since \( G \) is chordal, \( v_5 \) is not adjacent to \( v_6 \) and the graph in Fig. 8 is an induced subgraph of \( G \).

We consider two different cases.

**Case 1.** There is a path in \( G \) from \( v_5 \) to \( v_4 \) avoiding \( \{ v_1, v_2, v_3, v_4 \} \).

Let \( P : x_0 x_1 \ldots x_t \) with \( v_5 = x_0 \) and \( x_t = x_1 \) be a shortest such path. Since \( G \) is distance-hereditary, \( l = 3 \). Since \( G \) is chordal, \( N_C(v_2) \cap V_P = \{ x_t, x_{t+1}, \ldots, x_3 \} \) for some \( 1 \leq s \leq 3 \), \( N_C(v_3) \cap V_P = \{ x_t, x_1, \ldots, x_1 \} \) for some \( 0 \leq t \leq 2 \) and \( s \leq t \), i.e., \( 1 \leq s \leq t \leq 2 \).

If \( s = 2 \), then \( t = 2 \) and \( G[[v_2, v_3, v_5, x_1, x_2]] \) is a gem, which is a contradiction. Hence \( s = 1 \). If \( t = 1 \), then \( G[[v_2, v_3, v_5, x_1]] \) is a gem, which is a contradiction. Hence \( t = 2 \). If \( v_1 \) is not adjacent to \( x_1 \), then \( G[[v_1, v_2, v_3, v_5, x_1]] \) is a gem, which is a contradiction. Hence \( v_1 \) is adjacent to \( x_1 \). If \( v_1 \) is not adjacent to \( x_2 \), then \( G[[v_1, v_2, v_3, v_5, x_2]] \) is a gem, which is a contradiction. Hence \( v_1 \) is adjacent to \( x_2 \). If \( v_4 \) is not adjacent to \( x_1 \), then \( G[[v_1, v_2, v_3, v_4, v_5]] \) is a gem, which is a contradiction. Hence \( v_4 \) is adjacent to \( x_1 \). If \( v_4 \) is not adjacent to \( x_2 \), then \( G[[v_1, v_2, v_4, v_5, x_2]] \) is a gem, which is a contradiction. Hence \( v_4 \) is not adjacent to \( x_2 \). Now \( G[[v_1, v_2, v_3, v_4, v_5, x_1, x_2]] \) is isomorphic to \( F_3 \), which is a contradiction and the proof for this case is complete.

**Case 2.** There is no path in \( G \) from \( v_5 \) to \( v_4 \) avoiding \( \{ v_1, v_2, v_3, v_4 \} \).

Since \( G \) is 2-connected, there is a shortest path \( P \) in \( G \) from \( v_0 \) to a vertex in \( \{ v_1, v_3, v_4 \} \) avoiding \( v_2 \) and a shortest path \( Q \) in \( G \) from \( v_5 \) to a vertex in \( \{ v_1, v_2, v_4 \} \) avoiding \( v_5 \). In view of the assumption of Case 2, \( v_5 \) is not on \( Q \), and \( v_5 \) is not on \( P, P \) and \( Q \) have no interior vertex in common and there are no edges between interior vertices of \( P \) and interior vertices of \( Q \). Since \( G \) is chordal, \( v_2 \) is adjacent to every interior vertex of \( P \) and \( v_4 \) is adjacent to every vertex of \( Q \).

Since \( G \) is gem-free, \( Q \) has length exactly two. Let \( v_7 \) denote the interior vertex of \( Q \). If \( v_7 \) is adjacent either to \( v_1 \) or to \( v_2 \) but not to both, then \( G[[v_1, v_2, v_3, v_5, v_7]] \) is a gem, which is a contradiction. If \( v_7 \) is adjacent either to \( v_1 \) or to \( v_4 \) but not to both, then \( G[[v_1, v_2, v_3, v_5, v_7]] \) is a gem, which is a contradiction. Hence \( v_7 \) is adjacent to \( v_1 \) and \( v_2 \). Now we consider \( P \). Let \( v_8 \) denote the last interior vertex of \( P \). If \( v_8 \) is adjacent to \( v_4 \), then the chordality of \( G \) implies that \( v_8 \) is also adjacent to \( v_1 \) and \( v_2 \). Since \( G \) is gem-free, \( P \) has length exactly two. Hence if \( v_8 \) is adjacent to \( v_4 \), then
For any 2-connected distance-hereditary graph $G$ the following statements are equivalent.

(i) $G$ is a basic 5-leaf power;
(ii) $G$ is $(F_1, \ldots, F_8)$-free chordal;
(iii) $G$ is a plump dart or a plump bull or a 3-leaf power.

Proof. (i) $\Rightarrow$ (ii): Follows from Lemma 2 and the fact that 5-leaf powers are chordal.

(ii) $\Rightarrow$ (iii): Let $G$ satisfy (ii). If $G$ has a special homogeneous set, then $G$ is a plump dart by Lemma 3. Thus, we may assume that $G$ has no special homogeneous set. By Lemma 4, $cc(G)$ is a block graph. If $cc(G)$ is a tree, then $G$ is a 3-leaf power by Theorem 1 and we are done.

Hence we may assume that $cc(G)$ is not a tree. Note that, by definition, $cc(G)$ has no non-trivial critical clique and hence every endblock of $cc(G)$ is a clique of order 2. Hence $cc(G)$ must contain a block $B$ which is a clique of order at least 3 and which is not an endblock. Since $G$ is 2-connected, every cutvertex in $cc(G)$ corresponds to a clique of $G$ of order at least 2.

Let $x, y \in B$ be two distinct cutvertices of $cc(G)$. If $B$ contains a third cutvertex, then $G$ contains $F_7$ as an induced subgraph which is a contradiction. Hence every vertex in $B \setminus \{x, y\}$ is not a cutvertex in $cc(G)$. Since $B \setminus \{x, y\}$ is a critical clique in $cc(G)$, $|B \setminus \{x, y\}| \leq 1$.

Let $B$ and $B'$ be blocks of $cc(G)$ different from $B$ containing $x$ and $y$, respectively. If either $B'$ or $B''$ is not an endblock, then $G$ contains $F_8$ as an induced subgraph. Hence $B'$ and $B''$ are endblocks. Now $B$ together with every such pair $(B', B'')$ of blocks forms a bull. Thus $cc(G)$ is obtained from a bull by replacing the vertices of degree 1 with non-empty independent sets. Since $G$ is $F_9$-free, the cutvertices $x$ and $y$ stem from a $K_2$ in $G$. Altogether, we obtain that $G$ is a plump bull.

(iii) $\Rightarrow$ (i): Let $G$ satisfy (iii). First, if $G$ is a 3-leaf power with a 3-leaf root $T$, then the tree obtained from $T$ by subdividing each edge incident to a leaf of $T$ once yields a basic 5-leaf root for $G$.

If $G$ is a plump dart or plump bull, then $G$ is a basic 5-leaf power by Lemma 1 and the proof of Theorem 2 is complete.

5. Characterizing distance-hereditary (basic) 5-leaf powers

The gluing conditions for the blocks reflect the fact that all edges of a graph are entirely contained in one of its blocks. In order to avoid the creation of unwanted edges when composing basic 5-leaf roots of the blocks of a basic 5-leaf power $G$ to a basic 5-leaf root of $G$, we need to ensure appropriate distance conditions.

Definition 3. In a graph $G$, a vertex $v$ is called a special vertex if $N_G(v)$ is a clique module.

Lemma 5. Let $G = (V, E)$ be a 2-connected distance-hereditary basic 5-leaf power.

(i) Let $v$ be a vertex of $G$. If $G$ admits a basic 5-leaf root $T$ such that $d_T(v, x) \geq 5$ for all $x \in V \setminus \{v\}$, then $G$ is a 3-leaf power and $v$ is a special vertex of $G$.

(ii) Let $v_1$ and $v_2$ be two distinct vertices of $G$ and let $T$ be a basic 5-leaf root of $G$ such that $d_T(v_i, x) \geq 5$ for all $x \in V \setminus \{v_i\}$, $i = 1, 2$. Then $G = K_2$, or $v_1$ and $v_2$ are non-adjacent.

Proof. By Theorem 2, $G$ is either a plump dart or a plump bull or a 3-leaf power.

(i): Let $T$ be a basic 5-leaf root for $G$ such that $d_T(v, x) \geq 5$ for all $x \in V \setminus \{v\}$. It follows immediately with Lemma 1 that $G$ is a 3-leaf power.

First, assume that $N_G(v)$ is not a clique. Let $u_1, u_2 \in N_G(v)$ with $u_1u_2 \notin E_G$. By Observation 2, $u_1$ and $u_2$ have a common neighbor $w$ different from $v$. Since $G$ is chordal, $wv \in E_G$. Since $u_1$ and $u_2$ are not adjacent but have two common neighbors, $6 \leq d_T(u_1, u_2) \leq 7$. If $d_T(u_1, u_2) = 6$, then $d_T(v, u_1) = d_T(v, u_2) = 5$ uniquely determines the subtree of $T$ with the leaves $v, u_1, u_2$. Now $d_T(w, u_1), d_T(w, u_2) \leq 5$ implies $d_T(v, w) \leq 4$ which is a contradiction. If $d_T(u_1, u_2) = 7$, then we obtain $\{d_T(v, u_1), d_T(v, u_2)\} = \{4, 5\}$ which is a contradiction. Hence $N_G(v)$ is a clique.

Next, assume that $N_G(v)$ is not a module. Let $u_1, u_2 \in N_G(v)$ and $w \in V \setminus N_G[v]$ with $u_1w, u_2w \notin E_G$. By Observation 2, $v$ and $w$ have a common neighbor $u_3$ different from $u_1$. By the above, $\{u_1, u_2, u_3\}$ is a clique. Since $v$ and $w$ are not adjacent but have two common neighbors, $6 \leq d_T(v, w) \leq 7$.

If $d_T(v, w) = 6$, then $d_T(v, u_1) = d_T(v, u_3) \geq 5$ implies $\{d_T(w, u_1), d_T(w, u_3)\} = \{3, 5\}$ which uniquely determines the subtree of $T$ with the leaves $v, u_1, u_3$ up to exchanging the positions of $u_1$ and $u_3$. Now $d_T(v, u_2) = 5$ and $d_T(u_2, u_1), d_T(u_2, u_3) \leq 5$ implies $d_T(u_2, w) \leq 5$ which is a contradiction.

If $d_T(v, w) = 7$, then $d_T(v, u_1) = d_T(v, u_2) = 5$ implies $\max(d_T(w, u_1), d_T(w, u_3)) \geq 6$ which is a contradiction. Hence $N_G(v)$ is a clique module and $v$ is a special vertex of $G$.

(ii): Assume that $v_1$ and $v_2$ are adjacent. Then $d_T(v_1, v_2) = 5$. As $d_T(v_i, x) \geq 5$ for all $x \neq v_i, i = 1, 2, N_G(v_1) \cap N_G(v_2) = \emptyset$. This and the fact that $N_G(v_1)$ and $N_G(v_2)$ are cliques by (i) imply that $V_G = \{v_1, v_2\}$, i.e., $G$ is a $K_2$. □
Lemma 6. The graphs $F_9, \ldots, F_{34}$ in Figs. 9–11 are not basic 5-leaf powers.
Theorem 3. For any distance-hereditary graph $G$ the following statements are equivalent.

(i) $G$ is a basic 5-leaf power;
(ii) $G$ is $\{F_1, \ldots, F_{34}\}$-free chordal;
(iii) (a) The blocks of $G$ are 3-leaf powers, plump darts or plump bulbs.
    (b) For every two blocks $B \neq B'$ of $G$ with $B \cap B' = \{v\}$, if $B$ is not a 3-leaf power and $v$ is an inner vertex of $B$, then $B'$ is a 3-leaf power and $v$ is a special vertex of $B'$.
    (c) Every block that is a clique of order at least three contains at most one cut-vertex that is an inner vertex of another block.

Proof. (i) $\Rightarrow$ (ii): Follows from Lemmas 2 and 6 and the fact that 5-leaf powers are chordal.
(ii) $\Rightarrow$ (iii): Let $G$ satisfy (ii). Property (a) follows from Theorem 2 (part (ii) $\Rightarrow$ (iii)), property (b) follows from (a) and the fact that $G$ is $\{F_3, \ldots, F_{34}\}$-free, and property (c) follows from (a) and the fact that $G$ is $\{F_25, \ldots, F_{34}\}$-free chordal and that in a 2-connected chordal graph (a block of $G$) every edge belongs to a triangle.
(iii) $\Rightarrow$ (i): Let $G$ satisfy (iii). We first construct a basic 5-leaf root $T_B$ for each block $B$ of $G$, given by property (a), as follows.
If $B$ is a plump dart or plump bull, $T_B$ is as depicted in Figs. 5 and 6, respectively. In this case, $d_{T_B}(x,y) \geq 4$ for all distinct vertices $x,y$ in $B$ that are no inner vertices.
If $B$ is a 3-leaf power different from a clique of order at least three, let $T_B$ be as in the proof of Lemma 7. In this case, $d_{T_B}(x,y) \geq 4$ for all distinct vertices $x,y$ in $B$, and $d_{T_B}(x,y) \geq 5$ for all special vertices $x$ of $B$ and all $y \in B \setminus \{x\}$.
If $B$ is a clique with at least three vertices, then let $T_B$ be the star with leaf set $V_B$. If $B$ contains a cutvertex $v$ of $G$ that is an inner vertex of other blocks, then $T_B$ is obtained from $T_B$ by subdividing all edges not incident to $v$ once and the edge incident to $v$ two times. If $B$ contains no such cutvertices, then $T_B$ is obtained from $T_B$ by subdividing all edges once. Note that property (c) ensures that $T_B$ is a basic 5-leaf power of $B$. Moreover, $d_{T_B}(x,y) \geq 4$ for all distinct vertices $x,y$ in $B$, and if $B$ contains the cutvertex $v$ of $G$ that is an inner vertex of other blocks, $d_{T_B}(v,x) \geq 5$ for all $x \in B \setminus \{v\}$.

Next, we construct a basic 5-leaf root $T$ for $G$ from the $T_B$’s as follows. Consider each cutvertex $v$ of $G$. If $v$ belongs to exactly the blocks $B_1, \ldots, B_n$, then identify the neighbors of $v$ in the trees $T_{B_1}, \ldots, T_{B_n}$ and delete all but one copy of the leaf $v$ in the resulting graph. Since the block-cutvertex of any connected graph is a tree, the resulting graph is our tree $T$.

We claim that $T$ is a basic 5-leaf root of $G$. By construction, we have for every two distinct vertices $x,y$ of $G$ belonging to the same block $B$, $xy \in E_B$ if and only if $xy \in E_B$ if and only if $d_{T_B}(x,y) = d_{T_B}(x,y) \leq 5$. Thus, it remains to show that for all vertices $x,y$ of $G$ with $x \in B \setminus B, y \in B' \setminus B$ where $B$ and $B'$ are two distinct blocks of $G$, $d_{T_B}(x,y) \geq 5$. Clearly, by construction, we need only consider the case in which $B$ and $B'$ are non-disjoint blocks.
Let $B \cap B' = \{v\}$. By construction, $d_T(x,y) = d_{T_B}(x,y) + d_{T_B}(v,y) - 2$. Now, if $v$ is not an inner vertex neither of $B$ nor of $B'$, then $d_{T_B}(x,v) \geq 4$, $d_{T_B}(v,y) \geq 4$, hence $d_{T_B}(x,y) \geq 6$. If $v$ is an inner vertex, say, of $B$, then, by property (b), $B'$ is a 3-leaf power. In this case we have $d_{T_B}(x,u) \geq 3$, $d_{T_B}(v,y) \geq 5$, and hence again $d_{T_B}(x,y) \geq 6$. □

Since no graph in $\{F_25, \ldots, F_{34}\}$ is a proper induced subgraph of another graph in this set, these graphs together with the chordless cycles are exactly the minimal forbidden induced subgraphs for the class of distance-hereditary basic 5-leaf powers.

Let $F$ be the set of all minimal forbidden induced subgraphs different from the chordless cycles for the – not necessarily basic – distance-hereditary 5-leaf powers. We will now first describe $F$ abstractly and prove the correctness of this description. After that we will explain how $F$ can be obtained constructively in a very simple way from $\{F_1, \ldots, F_{34}\}$.

Let $F_0$ be the set of all graphs $G$ with the following three properties:
• G contains a graph from \{F_1, \ldots, F_{34}\} as an induced subgraph.
• G is chordal and distance-hereditary.
• G has no true twins.

Now let \mathcal{F} be the set of all graphs in \mathcal{F}_0 which are no proper induced subgraph of another graph in \mathcal{F}_0. From Theorem 3 we easily obtain the second main result of this section.

**Theorem 4.** For any distance-hereditary graph G the following statements are equivalent.

(i) G is a 5-leaf power;
(ii) G is \mathcal{F}-free chordal.

**Proof.** (i) ⇒ (ii): No graph in \mathcal{F} is a 5-leaf power, because, by definition, every 5-leaf root of it would necessarily contain a basic 5-leaf root for some graph in \{F_1, \ldots, F_{34}\}. Therefore, all 5-leaf powers are \mathcal{F}-free chordal.

(ii) ⇒ (i): Let G be distance-hereditary, \mathcal{F}-free and chordal. Let G' arise from G by contracting all pairs of true twins of G. Clearly, G' is distance-hereditary and chordal.

If G' would contain an induced subgraph F belonging to \{F_1, \ldots, F_{34}\}, then the possible true twins of F would not have been true twins within G. By the definition of \mathcal{F}, a minimal induced subgraph F' of G containing F as an induced subgraph with the property that no two vertices of F are true twins within F' would belong to \mathcal{F} which would contradict the assumption that G is \mathcal{F}-free.

Hence G' is \{F_1, \ldots, F_{34}\}-free and, by Theorem 3, G' has a basic 5-leaf root which easily yields a 5-leaf root of G. Hence G is a 5-leaf power. □

Before we give a constructive description of \mathcal{F}, we collect some observations: Those graphs in \{F_1, \ldots, F_{34}\} which do not contain true twins are also in \mathcal{F}, because every 5-leaf root for them would necessarily be basic. Furthermore, no graph in \mathcal{F} can have true twins, because otherwise it would not be minimal forbidden. This implies that the graphs \{F_1, \ldots, F_{34}\} which contain true twins cannot lie in \mathcal{F}. Similarly, none of the induced subgraphs of the graphs in \{F_1, \ldots, F_{34}\} can lie in \mathcal{F}, because the graphs F_1, \ldots, F_{34} are minimal forbidden. This implies that the graphs in \{F_1, \ldots, F_{34}\} which contain true twins are 5-leaf powers. Furthermore, it implies that all graphs in \mathcal{F} \setminus \{F_1, \ldots, F_{34}\} arise from the graphs in \{F_1, \ldots, F_{34}\} \setminus \mathcal{F} by adding further vertices and edges deleting their true twins. Therefore, \mathcal{F} can be constructed by the following steps starting with the empty set:

• Add to \mathcal{F} every graph among F_1, \ldots, F_{34} which has no true twins.
• For every graph \(F = (V_F, E_F) \in \{F_1, \ldots, F_{34}\}\) which has true twins add to \mathcal{F} all graphs which arise as follows:
  - Let \(\{x_1, y_1\}, \ldots, \{x_r, y_r\} \in E_F^2\) denote all pairs of true twins of \(F\).
  - Add r new vertices \(z_1, \ldots, z_r\).
  - For \(1 \leq i \leq r\) add exactly one of the two edges \(x_i z_i\) and \(y_i z_i\).
  - Arbitrarily identify new vertices which have the same neighbors within \(V_F\).
  - Arbitrarily add new edges among the remaining new vertices.
• Remove from \mathcal{F} all graphs which are
  - neither chordal,
  - nor distance-hereditary,
  - or a proper induced subgraph of another graph in \mathcal{F}.

6. Conclusion

With Theorems 3 and 4 we have given a characterization of distance-hereditary (basic) 5-leaf powers. This might be a first step towards the solution of the challenging problems of characterizing 5-leaf powers in general or distance-hereditary basic k-leaf powers for all \(k \geq 5\).

**References**