The Computational Complexity of Feedback Vertex Set, Hamiltonian Circuit, Dominating Set, Steiner Tree, and Bandwidth on Special Perfect Graphs 1)

By Andreas Brandstädt

Abstract: This paper gives a survey on the computational complexity of some NP-complete graph problems restricted to some special classes of perfect graphs. The graph problems are those given in the headline, and the graph classes include permutation graphs, bipartite permutation graphs, chordal bipartite graphs, strongly chordal graphs, interval graphs and similar classes.

Introduction

In order to overcome the intractability of NP-complete problems it is one of the main possibilities to restrict their instances to special classes and to see what structure properties of these special inputs help.

This is a wide field of research especially for graph problems and was surveyed in [14] for eleven basic graph problems and thirty graph classes. We want to restrict our survey to certain special classes of perfect graphs and the graph problems mentioned above where our survey differs from that of [14] mainly in two points: [14] does not include FEEDBACK VERTEX SET and BANDWIDTH which should be regarded as some of the most basic graph problems, and [14] does not include the class of chordal bipartite graphs. (This is a graph class which is already described in [12] — but of course there are many other important graph classes which cannot be included here. The main reason for mentioning this graph class are the results of [21].) Furthermore there are some new results found after [14] was published.

1. The graph classes

Let us start with the definition of the graph classes. Throughout this paper all graphs $G = (V, E)$ are finite, simple (i.e. without self-loops and multiple edges) and undirected. If $V' \subseteq V$ then let $G_{V'}$ denote the subgraph of $G$ induced by $V'$ i.e. $G_{V'} = (V', E')$ with $E' = \{ (u, v) : (u, v) \in E \text{ and } u, v \in V' \}$.

For notions which are not defined here (as e.g. path, cycle, chromatic number, clique etc.) cf. any standard textbook on graph theory (e.g. [12]).

Definition.

A graph $G$ is perfect if for every induced subgraph $G_{V'}$ of $G$ the chromatic number of $G_{V'}$ coincides with its maximum clique size. (Note that it is an interesting open problem whether perfect graphs can be recognized in polynomial time [14].)

1) Extended version of a lecture given at the Workshop "Mathematical Aspects of Informatics", Mügdegsprung (GDR), April 1986.
A. Brandstädt

\( G = (V, E) \) is a comparability graph (also frequently called transitively orientable) if there is an orientation \( F \) of the edges \( E \) (i.e. an assignment of directions to the edges of \( E \)) such that if \( (a, b) \in F \) and \( (b, c) \in F \) then also \( (a, c) \in F \).

\( G = (V, E) \) is a bipartite graph if \( G \) has chromatic number at most 2, i.e. \( V \) can be partitioned into at most two independent sets. For \( \bar{G} = (V, \bar{E}) \) let \( \bar{G} = (V, \bar{E}) \) denote the complement graph of \( G \), i.e. \( (u, v) \in \bar{E} \) iff \( (u, v) \notin E \).

\( G = (V, E) \) is a permutation graph if \( G \) and \( \bar{G} \) are comparability graphs. The name “permutation graph” comes from the definition via inversion graphs of permutations (cf. [12]).

A cycle \( C \) has a chord if two nonconsecutive vertices of \( C \) are joined by an edge. \( C = (v_1, \ldots, v_m) \) has an odd chord if there are nonconsecutive vertices \( v_i, v_j \) of \( C \) joined by a chord and there is an appropriate \( k \) with \( i + j = m + 1 \).

\( G = (V, E) \) is chordal bipartite if \( G \) is bipartite and each cycle of \( G \) of length at least 6 has a chord. (Note that cycles of length 4 have no chord in such graphs — chordal bipartite graphs are in general not chordal.)

Bipartite permutation graphs are just what is expressed by these two words — they are bipartite and permutation graphs.

\( G \) is chordal if each cycle of length greater than 3 has a chord.

\( G \) is a split graph if \( G \) and \( \bar{G} \) are chordal (for other characterizations see [12]).

\( G \) is strongly chordal if \( G \) is chordal and each cycle of even length greater than 4 has an odd chord. For other characterizations of this class introduced by Farber see [8].

\( G \) is a tree if \( G \) is connected and cycle-free.

The remaining classes are intersection graphs of certain models: vertices are corresponding to objects and there is an edge between two vertices iff the corresponding objects have a nonempty intersection. Note that chordal graphs are intersection graphs of subtrees of an appropriate tree.

\( G \) is an interval graph if \( G \) is the intersection graph of a set of intervals on the real line.

\( G \) is an undirected path graph if \( G \) is the intersection graph of a set of paths in a tree.
2. The problems

Let \( G = (V, E) \) be a graph. The set \( F \subseteq V \) is a feedback vertex set (fvs) of \( G \) if \( G_{V - F} \) is cycle-free i.e. \( F \) contains at least one vertex from each cycle in \( G \).

**Feedback Vertex Set (FVS)**

\[
(G, k) : G = (V, E) \text{ is a graph and } k \text{ a positive integer and there is an fvs } F \subseteq V \text{ of } G \text{ with } |F| \leq k
\]

is known as an NP-complete problem even for bipartite graphs — in [11] it is formulated for directed graphs under [GT 7].

A graph \( G = (V, E) \) has a Hamiltonian circuit if there is a cycle of \( G \) containing each vertex of \( V \).

**Hamiltonian Circuit (HC)**

\[
(G, k) : G = (V, E) \text{ is a graph which has a Hamiltonian circuit}
\]

is known to be NP-complete even for bipartite and split graphs and many other restrictions (cf. [14]).

A subset \( D \subseteq V \) of a graph \( G = (V, E) \) is a dominating set of \( G \) if for all \( u \in V - D \) there is a \( v \in D \) with \( (u, v) \in E \).

**Dominating Set (DS)**

\[
(G, k) : G = (V, E) \text{ is a graph and } k \text{ a positive integer and there is a dominating set } D \subseteq V \text{ of } G \text{ with } |D| \leq k
\]

([GT 2] of [11]) is known to be NP-complete even for bipartite graphs.

Note that also variants of DS (where the dominating set \( D \) is independent or connected or a clique or \( G_D \) has no isolated vertices) were investigated (cf. [13], [4], [7], [24]).

The minimum cardinality Steiner tree problem is the problem to connect (if possible) a given set \( T \subseteq V \) (of target vertices) by adding a minimum number of vertices to \( T \).

**Steiner Tree (ST)**

\[
(G, T, k) : G = (V, E) \text{ is a graph and } k \text{ a positive integer and } T \subseteq V \text{ and there is an } S \subset V \text{ with } |S| \leq k \text{ and } G_{T, S} \text{ is connected}
\]

is known to be NP-complete.

Note that the Steiner tree problem is usually formulated for edge weighted graphs and in the case of unweighted graphs edge and vertex minimization versions are equivalent.

A (linear) layout of \( G = (V, E) \) is a one-to-one correspondence \( L \) assigning a unique integer to each vertex from \( V \).

The bandwidth of \( G \) with respect to the layout \( L \) is

\[
b(G, L) = \max \{ |L(u) - L(v)| : \langle u, v \rangle \in E \}.
\]

The bandwidth of \( G \) is

\[
b(G) = \min \{ b(G, L) : L \text{ is a layout of } G \}.
\]

**Bandwidth (BW)**

\[
(G, k) : G = (V, E) \text{ is a graph and } k \text{ a positive integer and } b(G) \leq k
\]
is known to be NP-complete even for special trees (caterpillars with hair length 3 — cf. [19]).

The following table shows the complexity of the problem $X$ restricted to the graph class $Y$. Hereby $P$ denotes solvability in polynomial time, $N$ denotes NP-completeness, \( \dagger \) denotes that the complexity of the problem is unknown. There are many entries within the table which are easy consequences of other ones or are trivial. Only for the basic ones we indicate the source references. (33 of the 65 entries occur already in [14].)

<table>
<thead>
<tr>
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<th>FVS</th>
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<th>DS</th>
<th>ST</th>
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</table>

### 3. The basic complexity results

#### 3.1. FVS

By a simple reduction from VERTEX COVER [GT 1] [11]) it can be shown that FVS is NP-complete for bipartite graphs. The basic results are:

1. **FVS is solvable in polynomial time for permutation graphs** (by a dynamic programming approach which also uses structure properties of bipartite permutation graphs).
2. **FVS is solvable in polynomial time for strongly chordal graphs**, [5] where a characterization of strongly chordal graphs given by Farber [5] is used which shows that a graph is strongly chordal if it has a so-called simple elimination scheme — a concept which is very similar to that of simplicial vertices for chordal graphs — cf. [12].

For split graphs it is easy to show that $FVS \in P$. All other $N$- or $P$-entries in the first column follow by inclusions shown in the figure (p. 472).

\( \dagger \) see note on p. 477
3.2. HC

There are few classes for which the HC problem is solvable in polynomial time. Among them there are classes for which the answer is always NO (e.g. for trees) or always YES. This is not the case for bipartite permutation and interval graphs. The basic results in column 2 are:

1. HC is solvable in polynomial time for interval graphs — a result of Keil [16] which can be extended to directed path graphs ([18]) and makes use of the clique structure of the intersection model of those graphs.
2. HC is solvable in polynomial time for bipartite permutation graphs ([3], [23]) which use special structure properties of such graphs (the existence of a so-called strong ordering of the vertices).

By results of Nash-Williams [22] and Karp [15] it follows that HC is NP-complete for bipartite graphs (and therefore also for split graphs). Meanwhile many restrictions are known where HC remains NP-complete (cf. [14]). A further one is

3. HC remains NP-complete for chordal bipartite graphs ([20]).

3.3. DS

Dominating set together with its several variations was entitled "the world's embroidery champion" in [13]. We restrict our survey to DS itself. The main results are

1. DS is solvable in polynomial time for permutation graphs (by the use of dynamic programming), [10], [3], [4].
2. DS is NP-complete for chordal bipartite graphs (by a reduction from VERTEX COVER using a local replacement for vertices and edges [21]).
3. DS is NP-complete for split ([6]) and undirected path graphs ([2]) and
4. DS is solvable in polynomial time for strongly chordal graphs, [9] (for the unweighted version it can again be done via a simple elimination ordering of the graph).

3.4. ST

The Steiner tree problem's complexity is very similar to that of DS. In [24] it is observed that whenever the complexities of the variant of DS where the dominating set is connected and of ST are known they are the same.

The columns for DS and ST also coincide in all positions with the exception that for ST and undirected path graphs the complexity is unknown. The main results for ST are

1. ST is solvable in polynomial time for permutation graphs, [7] (by using structural properties of an "intersection model" for permutation graphs).
2. ST is NP-complete for chordal bipartite graphs (which answers a question of G. Ausiello asked at FCT '83 motivated by database problems ([21]) — the reduction is very similar to that one for DS).
3. ST is NP-complete for split graphs, [24].
4. ST is solvable in polynomial time for strongly chordal graphs, [24].
3.5. BW

BANDWIDTH is one of the few natural problems which remain NP-complete even for trees. The main results are

1. BW is NP-complete even for very special trees — so-called caterpillars with hair length 3 ([10]) whereas it is known that for caterpillars of hair length at most 2 BW is solvable in polynomial time ([17]).

2. BW is solvable in polynomial time for interval graphs ([17]). In [14] it is mentioned that the author does not know any NP-completeness results for interval graphs and BW is mentioned as a possibility for such a result. [17] shows that this is not the case.

It would be interesting to know the complexity of BW for split or permutation graphs.

References

The Computational Complexity


Résumé

В статье дан обзор вычислительной сложности некоторых NP-полных проблем для некоторых классов совершенных графов. Рассмотрены следующие проблемы: множество вершин, разрезавших контуры; гамильтонов цикл; доминирующее множество; дерево Штейнера; ширина графа. Изучены следующие специальные классы графов: перестановочные графы, двудольные перестановочные графы, хордально-двудольные графы, сильно-хордальные графы, интервалльные графы и т.п.

Kurzfassung


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Author's address:

Dr. A. Brandstättd
Sektion Mathematik der
Friedrich-Schiller-Universität Jena
6900 Jena
German Democratic Republic

Note added in proof: Recently Th. Guntermann has shown that FVS in solvable in polynomial time for chordal (and hence for undirected path) graphs.
Book Review


"The idea for this book grew out of the successful workshop 'Cycles in Graphs 1982' held at Simon Fraser University 5 July—20 August 1982. The primary purpose of the workshop was to gather the world's leading experts on cycles in graphs and have them discuss what they felt were the outstanding problems in the area... The result is this volume... It should not be viewed as a conference proceedings... The papers present original research for the most part as there are three survey papers among the forty-five papers in the volume..." (aus dem Vorwort).

An den 45 Beiträgen sind 59 Autoren beteiligt, der Umfang der einzelnen Beiträge ist sehr unterschiedlich. Die 3 Übersichtsartikel sind:

- F. Jaeger (IMAG, Grenoble): A Survey of the Cycle Double Cover Conjecture (12 S.);
- J. A. Bondy (University of Waterloo, Canada): Kotzig's Conjecture on Generalized Friendship Graphs (16 S.);
- R. Duke (Georgia Institute of Technology): Types of Cycles in Hypergraphs (19 S.).

Die einzelnen Arbeiten gruppieren sich etwa um folgende Themenkreise:

- The double cover conjecture
- Hamilton circuits in (Cayley graphs and digraphs, in transitive graphs, in direct products of graphs, in 3-connected cubic maps, in tournaments, in random graphs)
- Pancyclicity
- Longest cycles and paths
- Cycles with given length and other given properties
- Cycle decompositions
- Algorithms for finding cycles with specified properties
- Generalized friendship graphs
- Cycle basis problems
- Cycles in hypergraphs
- Special problems concerning cycles and paths

Den Abschluß bildet eine Liste von 42 offenen Problemen.

Es ist zu bedenken, daß einige Graphentheoretiker der alten Welt, die durch Pionierarbeit und Buchpublikationen über Kreise in Graphen bekannt geworden sind, nicht zu den Autoren des Bandes gehören.

Das Werk vermittelt einen ausgezeichneten Überblick über Schwerpunkte gegenwärtiger Forschung in allen Fragen, die Kreise in Graphen betreffen. H. Sachs