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Note

# On stable cutsets in line graphs<sup>☆</sup>

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## Abstract

We answer a question of Brandstädt et al. by showing that deciding whether a line graph with maximum degree 5 has a stable cutset is **NP**-complete. Conversely, the existence of a stable cutset in a line graph with maximum degree at most 4 can be decided efficiently. The proof of our **NP**-completeness result is based on a refinement on a result due to Chvátal that recognizing decomposable graphs with maximum degree 4 is an **NP**-complete problem. Here, a graph is decomposable if its vertices can be colored red and blue in such a way that each color appears on at least one vertex but each vertex  $v$  has at most one neighbor having a different color from  $v$ . We also discuss some open problems on stable cutsets.

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## 1. Introduction

In a graph, a *stable set* (a *clique*) is a set of pairwise non-adjacent (adjacent) vertices. A *cutset* (or *separator*) of a graph  $G$  is a set  $S$  of vertices such that  $G - S$  is disconnected. A *stable cutset* (a *clique cutset*) is a cutset which is also a stable set (a clique).

Clique cutsets are a well-studied kind of separators in the literature, and have been used in divide-and-conquer algorithms for various graph problems, such as graph

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coloring and finding maximum stable sets; see [12]. Applications of clique cutsets in algorithm designing based on the fact that clique cutsets in arbitrary graphs can be found in polynomial time [14].

The importance of stable cutsets has been demonstrated first in [4,13] in connection to perfect graphs. Tucker [13] proved that if  $S$  is a stable cutset in  $G$  and if no induced cycle of odd length  $\geq 5$  in  $G$  has a vertex in  $S$  then the coloring problem on  $G$  can be reduced to the same problem on the smaller subgraphs induced by  $S$  and the components of  $G - S$ . Unfortunately, deciding whether a graph has a stable cutset is **NP**-complete (let **STABLE CUTSET** denote this problem). This is an easy consequence of a result due to Chvátal [3] on decomposable graphs; see also [1,7]. In fact, Chvátal's results from [3] implies the following.

**Theorem 1.** **STABLE CUTSET** is **NP**-complete, even if the input graph is restricted to line graphs with maximum degree 6.

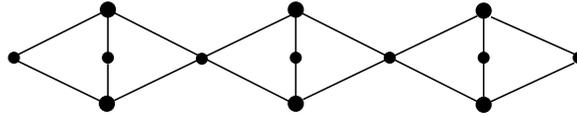
In [1], Brandstädt et al. then asked whether the degree constraint in Theorem 1 is best possible. In this paper, we consider this question and prove that **STABLE CUTSET** is **NP**-complete, even for 5-regular line graphs of bipartite graphs, and is efficiently solvable for line graphs with maximum degree 4. Also, some open problems will be discussed.

Let  $G$  be a graph. The vertex set and the edge set of  $G$  is denoted by  $V(G)$  and  $E(G)$ , respectively. The neighborhood of a vertex  $v$  in  $G$ , denoted by  $N_G(v)$ , is the set of all vertices in  $G$  adjacent to  $v$ ; if the context is clear, we simply write  $N(v)$ . For a subset  $S \subseteq V(G)$ ,  $N_S(v)$  stands for  $N(v) \cap S$ .

## 2. Decomposable graphs revisited

A graph is *decomposable* if its vertices can be colored red and blue in such a way that each color appears on at least one vertex but each vertex  $v$  has at most one neighbor having a different color from  $v$ . In other words, a graph is decomposable if its vertices can be partitioned into two non-empty parts such that the edges connecting vertices of different parts form an induced matching. Below we will use the fact that the complete bipartite graph  $K_{2,3}$  is indecomposable. This is because in *any* coloring of its vertices in red and blue, at least two of the degree 2-vertices must have the same color, say red. Now, if one degree 3-vertex is colored by blue, it has two red neighbors. If both the degree 3-vertex are colored by red then all vertices are red or the third degree 2-vertex is blue and has two red neighbors.

Chvátal [3] proved that recognizing decomposable graphs is **NP**-complete, even for graphs with maximum degree 4. Based on Chvátal's original proof we extend this result. Thereby, we also extend a result of Moshi [10] that recognizing decomposable graphs is **NP**-complete, even if the input is restricted to bipartite graphs of minimum degree 2.

Fig. 1. The graph  $A_v$  ( $d=1$ ).

**Theorem 2.** *Recognizing decomposable graphs is NP-complete, even if the input is restricted to bipartite graphs with one color class consisting only of vertices of degree 3 and the other color class consisting only of vertices of degree 4.*

**Proof.** A hypergraph is called *bicolorable* if its vertices can be colored red and blue in such a way that no edge is monochromatic. Lovász [9] proved that recognizing bicolorable hypergraphs is an NP-complete problem, even if the input is restricted to hypergraphs with each edge having size 3. Given such a hypergraph  $H$  we shall construct a graph  $G$  such that  $G$  is decomposable if and only if  $H$  is bicolorable. Now observe that the complete bipartite graph  $K_{2,3}$  is indecomposable. For each vertex  $v$  of  $H$ , belonging to  $d$  edges, take a chain  $A_v$  of  $4d-1$  different complete bipartite graphs  $K_{2,3}$ , say  $D_1, \dots, D_{4d-1}$  each with the partite set  $C_j = \{a_j, b_j, c_j\}$  of degree two vertices for  $j=1, \dots, 4d-1$ , such that we identify  $c_j$  with  $a_{j+1}$  for  $j=1, \dots, 4d-2$ . ( $A_v$  is illustrated in Fig. 1 for  $d=1$ ). Label one of the vertices of degree 2 in this  $A_v$  by  $v^*$  and the remaining  $4d$  vertices of degree two by the distinct labels  $(v, e, i)_{(j)}$  with  $i=0, 1, j=1, 2$  and  $e$  running through all the edges with  $v \in e$ .

For each  $i=0, 1$  take a similar chain  $B_i$  of  $2n+m-2$  complete bipartite graphs  $K_{2,3}$  with  $n$  and  $m$  standing for the number of vertices and edges in  $H$ , respectively. Label  $n$  of the vertices of degree two in  $B_i$  by  $(v, i)_1$  and  $n$  of the vertices of degree two in  $B_i$  by  $(v, i)_2$  with  $v$  running through all the vertices of  $H$  and label the remaining  $m$  vertices of degree two in  $B_i$  by  $(e, i)$  with  $e$  running through all the edges of  $H$ . For each  $i=0, 1$  and for each edge  $e = \{x, y, z\}$  of  $H$  add the graph  $A_{e,i}$  shown in Fig. 2 in which the vertices of degree two are labeled.

Furthermore, for each vertex  $v$  of  $H$ , add the graph  $B_v$  shown in Fig. 3 in which the vertices of degree two are labeled.

Note that in each graph described above, vertices of degree 2 are exactly the labeled vertices, and that each graph is bipartite with one color class of the bipartition consisting only of vertices of degree 3 and the other color class consisting only of vertices of degree 2 or 4.

Now, identify vertices with the same labels. The resulting graph is the desired  $G$ , a bipartite graph with minimum degree 3, maximum degree 4, all vertices of maximum degree 4 form a stable set and likewise all vertices of minimum degree 3 form another stable set.

Now consider a decomposition of  $G$ . As each complete bipartite graph  $K_{2,3}$  is indecomposable, the  $n$  graphs  $A_v$  and the two graphs  $B_i$  are indecomposable: in each of them, all the vertices have the same color. Without loss of generality, say that all the vertices of  $B_0$  are red. Now all the vertices in  $B_1$  must be blue: otherwise all  $A_v$  would be red [since the (red colored) graph  $B_v$  in Fig. 3 would force red on  $v^*$ ] and then it is

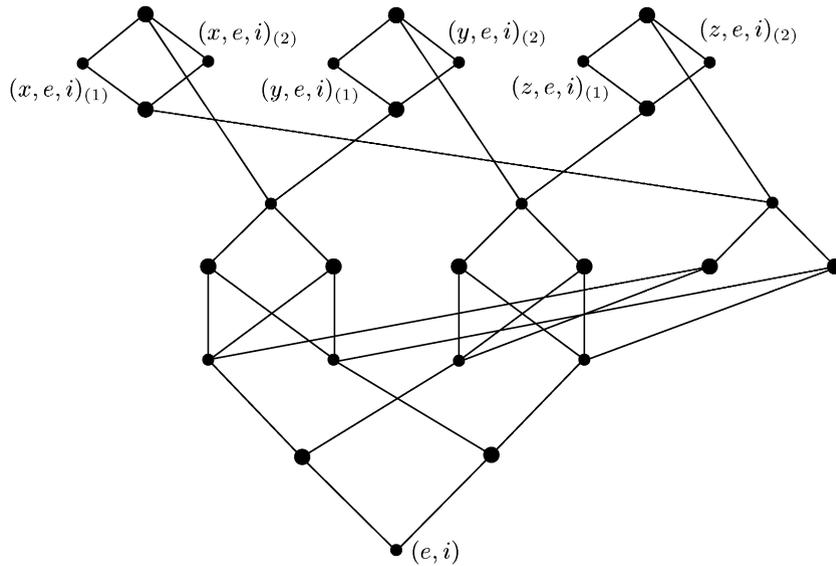


Fig. 2. The graph  $A_{e,i}$ .

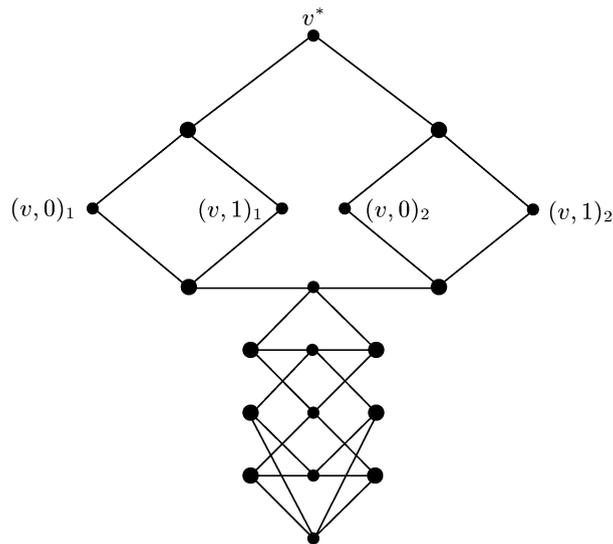


Fig. 3. The graph  $B_v$ .

not difficult to observe that all vertices of the graphs  $A_{e,i}$  of Fig. 2 would receive the color red as well and therefore the entire graph would be red, a contradiction. Next, consider an arbitrary edge  $e = \{x, y, z\}$  of  $H$ . At least one of the graphs  $A_x, A_y$  or  $A_z$  must be red [otherwise the graph  $A_{e,0}$  in Fig. 2 would force blue on  $(e, 0)$ ] and at

least one of them must be blue [otherwise the graph  $A_{e,1}$  in Fig. 2 would force red on  $(e, 1)$ ]. Hence, the colors on the  $n$  graphs  $A_v$  define a bicoloring of  $H$ . Conversely, each bicoloring of  $H$  may be used to color the  $n$  graphs  $A_v$  and then extended into a decomposition of  $G$ .  $\square$

**Remark 3.** It should be remarked that we could also use graphs of Fig. 1 of ‘arbitrary’ length. Hence, for every given  $\delta > 0$  we can choose one color class containing all vertices of degree 3 and the other color class containing vertices of degree 2 and 4 s.t. the average degree is at most  $(1 + \delta)3$ .

As Chvátal has shown, the **NP**-result is best possible with respect to degree constraints:

**Theorem 4** (Chvátal [3]). *Decomposable graphs of maximum degree 3 can be recognized in polynomial time.*

We will make use of this theorem in proving that **STABLE CUTSET** can be solved in polynomial time for line graphs of maximum degree at most 4.

### 3. Stable cutsets in line graphs

As suggested by Cunningham (see [1]) and shown by Brandstädt et al. the **STABLE CUTSET** problem for line graphs can be derived by the result of Chvátal on decomposable graphs.

Recall that the *line graph*  $L(G)$  of a graph  $G$  has the edges of  $G$  as its vertices, and two distinct edges of  $G$  are adjacent in  $L(G)$  if they are incident in  $G$ . For a given line graph  $L$ , the root graph  $G$  of  $L$ , that is  $L = L(G)$ , can be determined in linear time (see [5,8,11]).

The relationship between decomposability and a stable cutset is

**Proposition 5** (Brandstädt et al. [1]). *If  $L(G)$  has a stable cutset, then  $G$  is decomposable. If  $G$  is decomposable and has minimum degree at least 2, then  $L(G)$  has a stable cutset.*

Now observe that if  $G$  is a bipartite graph with one color class consisting only of vertices of degree 3 and the other color class consisting only of vertices of degree 4, then  $L(G)$  is a 5-regular graph. With this last proposition, the extension of the complexity result of Chvátal as proven in the last section and the latter observation we derive the following strengthening of the **NP**-complete result given in Theorem 1.

**Theorem 6.** **STABLE CUTSET** is **NP**-complete, even if the input is restricted to 5-regular line graphs of bipartite graphs.

By Remark 3 at the end of the last section we also obtain the following related result.

**Theorem 7.** STABLE CUTSET is **NP**-complete, even if the input is restricted to line graphs of order  $n$  and size  $(2 + \varepsilon)n$  of bipartite graphs with minimum degree at least 2, maximum degree 4, all vertices of degree 3 form an independent set and likewise all vertices of even degree form an independent set.

These results are best possible, since we will demonstrate in the following that STABLE CUTSET is polynomial if the input is restricted to line graphs with maximum degree at most 4.

#### 4. Observations

As a corollary of Theorem 4 and Observation 10 below, STABLE CUTSET can be solved in polynomial time on line graphs of maximum degree 3. We will see in the next sections that STABLE CUTSET can be solved in polynomial time for arbitrary graphs of maximum degree 3 and for line graphs of maximum degree 4. For this purpose, we need some observations.

**Observation 8.** Let  $S$  be a stable cutset of  $G$  and let  $v \in V(G)$ . If  $N(v)$  is not a stable set, then  $G - v$  has a stable cutset.

**Proof.** If  $v \in S$ ,  $S - v$  is a stable cutset in  $G - v$ . If  $v$  belongs to component  $A$  of  $G - S$ , then, as  $N(v)$  is not a stable set,  $|A| \geq 2$ , hence  $S$  is a stable cutset in  $G - v$ .  $\square$

**Observation 9.** Let  $v \in V(G)$ , and let  $S$  be a stable cutset of  $G - v$ . If  $S \cap N(v) = \emptyset$  or  $N(v) - S$  belongs to the same component in  $(G - v) - S$ , then  $G$  has a stable cutset.

**Proof.** In the first case,  $S \cup \{v\}$  is a stable cutset in  $G$ , in the second case,  $S$  is a stable cutset in  $G$ .  $\square$

A vertex  $v$  in  $G$  is *simplicial* if  $N_G(v)$  is a clique. The observation below directly follows from the previous two.

**Observation 10.** Assume that  $G$  has at least three vertices, and let  $v$  be a simplicial vertex of  $G$ . Then  $G$  has a stable cutset if and only if  $G - v$  has a stable cutset.

The following observation admits an argument that will be used in further discussions.

**Observation 11.** Let  $v$  be a vertex of a graph  $G$  such that  $N(v)$  can be partitioned into a clique  $C$  and a vertex  $a$ . Suppose that  $G - v$  has a stable cutset  $S$ . Then  $G$  has a stable cutset, or we may assume that there exists a vertex  $c \in C$  such that

- $S \cap N(v) = \{c\}$ ,  $N(a) \cap C \subseteq \{c\}$ ,
- $(G - v) - S$  has exactly two components, and  $a$  and  $C - \{c\}$  belong to different components of  $(G - v) - S$ .

**Proof.** If  $S \cap N(v) = \emptyset$  then, by Observation 9,  $G$  has a stable cutset. So, we may assume that  $S \cap N(v) \neq \emptyset$ . If  $a \in S$  then  $N(v) - S \subseteq C$ , hence by Observation 9 again,  $G$  has a stable cutset. Thus, we may assume that  $a \notin S$ , hence  $S$  meets  $N(v)$  in (exactly) one vertex  $c \in C$ :

$$S \cap N(v) = \{c\}.$$

If  $N(a) \cap C \not\subseteq \{c\}$  (in particular  $N(a) \cap C \neq \emptyset$ ) then  $N(v) - S = N(v) - \{c\}$  induces a connected subgraph in  $G - v$ , and by Observation 9 again,  $G$  has a stable cutset. Thus, we may assume that

$$N(a) \cap C \subseteq \{c\}.$$

If  $N(v) - \{c\}$  belongs to the same component in  $(G - v) - S$  then, by Observation 9,  $G$  has a stable cutset. Thus we may assume that

$$a \text{ and } C - \{c\} \text{ belong to different components of } (G - v) - S.$$

Let  $A$  and  $B$  be the components of  $(G - v) - S$  with  $a \in A$  and  $C - \{c\} \subseteq B$ . If, finally,  $(G - v) - S$  has a third component different from  $A$  and  $B$  then clearly  $S$  is a stable cutset in  $G$  disconnecting  $v$  and this third component. Thus we may assume that

$$(G - v) - S \text{ has exactly two components.}$$

The observation follows.  $\square$

### 5. Stable cutsets in graphs with maximum degree 3

We show in this section that STABLE CUTSET can be solved in polynomial time on (not necessarily line) graphs with maximum degree at most 3. We discuss the non-trivial case where the input graph  $G$  has maximum degree 3 and at least seven vertices. Furthermore, we may assume that every vertex of  $G$  has a non-stable neighborhood (otherwise we are done), and that  $G$  has no simplicial vertex (by Observation 10).

Consider a vertex  $v$  of maximum degree 3. We will show that  $G$  has a stable cutset if and only if  $G - v$  has a stable cutset. By Observation 8 we need only to show the if part. Thus, let  $S$  be an inclusion-minimal stable cutset in  $G - v$ . We will show that  $G$  has a stable cutset.

As  $N(v)$  is not a stable set, it consists of three vertices,  $a, b, c$  such that  $b$  and  $c$  are adjacent, say. By Observation 11, we may assume that  $S \cap N(v) = \{c\}$ ,  $N(a) \cap \{b, c\} \subseteq \{c\}$ , and  $(G - v) - S$  has exactly two components. Let  $A, B$  be the components of  $(G - v) - S$  with  $a \in A$  and  $b \in B$ .

*Case 1:*  $a$  is adjacent to  $c$ .

If  $A \neq \{a\}$  then the neighbor of  $a$  in  $A - \{a\}$  and the vertex  $b$  form a stable cutset in  $G$  disconnecting  $a$  and the neighbor of  $b$  different from  $v$  and  $c$  (such a neighbor of  $b$  exists because  $b$  is not simplicial). Similarly, if  $B \neq \{b\}$ ,  $G$  has a stable cutset. Thus we may assume that  $A = \{a\}$  and  $B = \{b\}$ . Then, as  $G$  has at least 5 vertices,  $|S| \geq 2$ , hence  $\{a, b\}$  is a stable cutset in  $G$  disconnecting  $v$  and  $S - \{c\}$ . Case 1 is settled.

Case 2:  $a$  is non-adjacent to  $c$ .

By the minimality of  $S$ ,  $c$  has (exactly one) neighbor  $c' \in A$ . If  $a$  has no neighbor in  $S$ ,  $S \cup \{a\}$  is a stable cutset in  $G$  disconnecting  $c'$  and  $v$ . So, let us assume that  $a$  has (exactly one) neighbor  $a' \in S$ . By the minimality of  $S$ ,  $a'$  has a neighbor  $a'' \in B$ .

If  $a'' \neq b$ , then  $b$  has exactly one neighbor  $b' \in B - \{b\}$ , and  $\{a, c, b'\}$  is a stable cutset in  $G$  disconnecting  $b$  and  $c'$ . So, we may assume that  $a'' = b$ , hence  $B = \{b\}$ . Now, if  $a'$  and  $c'$  are non-adjacent, then  $\{a', c', v\}$  is a stable cutset in  $G$  disconnecting  $a$  and  $b$ , and if  $a$  and  $c'$  are non-adjacent, then  $\{a, b, c'\}$  is a stable cutset in  $G$  disconnecting  $v$  and  $a'$ . The case  $a'c'$  and  $ac'$  are edges of  $G$  cannot occur because  $G$  has at least seven vertices. Case 2 is settled.

Thus, for every vertex  $v$  of maximum degree 3,  $G$  has a stable cutset if and only if  $G - v$  has a stable cutset. Repeat the same reduction on  $G' = G - v$  if  $G'$  still has maximum degree 3 and at least seven vertices. In this way, we can determine in polynomial time whether  $G$  has a stable cutset. Therefore, we obtain

**Theorem 12.** STABLE CUTSET can be solved in polynomial time on graphs with maximum degree at most 3.

## 6. Stable cutsets in line graphs of maximum degree 4

Let  $L = L(G)$  with maximum degree  $\Delta = \Delta(L) = 4$  and at least 13 vertices. We shall reduce  $L$  to a line graph  $L^*$  such that  $L^* = L(G^*)$ , and

- (i)  $L$  has a stable cutset if and only if  $L^*$  has a stable cut set, and
- (ii)  $G^*$  has maximum degree at most 3 and minimum degree 2.

Thus, by Proposition 5 and Theorem 4, we obtain

**Theorem 13.** STABLE CUTSET can be solved in polynomial time on line graphs of maximum degree at most 4.

*The reduction:* First, by Observation 10, we may assume that  $L$  has no simplicial vertex. Let  $v$  be a vertex of  $L$  of maximum degree  $d(v) = 4$ , and let  $xy$  be the corresponding edge in  $G$ . We call  $v$  *bad* if (exactly) one of  $x, y$  is of degree 4 in  $G$  ( $x$  or  $y$  cannot have degree 1 in  $G$  otherwise  $v$  would be a simplicial vertex in  $L$ ). If no vertex of degree 4 in  $L$  is bad, then  $G$  has maximum degree 3 and we are done by Theorem 4. If  $L$  has a bad vertex  $v$  of degree 4 then we will show that  $L$  has a stable cutset if and only if  $L - v$  has a stable cutset. Since  $L - v$  is again a line graph, we obtain in this way the desired line graph  $L^*$ . Let  $v$  be a bad vertex of maximum degree 4 in  $L$ , and let  $xy$  be the corresponding edge in  $G$ , with say  $d_G(x) = 4$ . By Observation 8, we need only show that if  $L - v$  has a stable cutset, then  $L$  has a stable cutset. Thus, let  $S$  be an inclusion-minimal stable cutset in  $L - v$ .

We know that three vertices in  $N(v)$  (which correspond to the three edges in  $G$  at  $x$ , different from  $xy$ ), say  $v_1, v_2, v_3$ , induce a triangle. Let  $w$  denote the fourth vertex of  $N(v)$ . By Observation 11, we may assume that  $S \cap N(v) = \{v_1\}$ ,  $N(w) \cap \{v_1, v_2, v_3\} \subseteq \{v_1\}$ , and  $(L - v) - S$  has exactly two components. Let  $A, B$  be the components of

$(L - v) - S$  with  $w \in A$  and  $\{v_2, v_3\} \subseteq B$ . Since  $S$  is minimal,

$v_1$  has (exactly one) neighbor  $v'_1 \in A$ ;

possibly  $v'_1 = w$ . Let  $B' = B - \{v_2, v_3\}$ .

We call a set of vertices in  $L$  *forbidden* if it induces a forbidden subgraph for line graphs (such as a  $K_{1,3}$  and so on; see [6] for a survey on line graphs).

**Claim 1.** If  $B' \neq \emptyset$ , then no vertex in  $B'$  is adjacent to both  $v_2$  and  $v_3$ .

(Proof: If  $b' \in B'$  is adjacent to both  $v_2$  and  $v_3$  then a neighbor of  $b'$  different from  $v_2$  and  $v_3$ , which exists because  $b'$  is not simplicial, together with  $b', v_1, v_2, v_3$  and  $v$  would form a forbidden set).

Now, assume that  $w$  is adjacent to  $v_1$ . (That is,  $v'_1 = w$ .)

Then  $A$  consists of exactly  $w$ . Otherwise a neighbor of  $w$  in  $A$  together with  $v, v_1, v_2, v_3$  and  $w$  would form a forbidden set. Moreover,  $w$  has no neighbor in  $S - \{v_1\}$ : If  $s' \in S - \{v_1\}$  is adjacent to  $w$  then  $s', w, v, v_1, v_i$  (if  $s'$  is adjacent to  $v_i \in \{v_2, v_3\}$ ) or  $s', w, v, v_1, v_2, v_3$  (otherwise) form a forbidden set. Therefore,  $w$  is a simplicial vertex, a contradiction. Thus we may assume that

$w$  is non-adjacent to  $v_1$ .

If  $w$  has no neighbor in  $S$  then  $S \cup \{w\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $v'_1$ . Thus we may assume that

$w$  has a neighbor  $w' \in S$ .

We now distinguish two cases.

*Case 1:*  $w'$  and  $v'_1$  are non-adjacent.

If  $w$  is adjacent to  $v'_1$ , then  $w$  and  $v, w', v'_1$  induce a  $K_{1,3}$ , a contradiction. Thus, we may assume that

$w$  is non-adjacent to  $v'_1$ .

If  $B' \neq \emptyset$  then, without loss of generality, let  $v'_2$  be the neighbor of  $v_2$  in  $B'$ . By Claim 1,  $v'_2$  is non-adjacent to  $v_3$ , hence  $\{v'_2, v_3, w, v'_1\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $w'$ . Thus, we may assume that

$B = \{v_2, v_3\}$ .

Then, by the minimality of  $S$ ,  $w'$  must be adjacent to  $v_2$  or  $v_3$ , but not both (if  $w'$  is adjacent to both  $v_2$  and  $v_3$  then  $\{w', v_2, v_3, v, w\}$  is a forbidden set). Let, without loss of generality,

$w'v_2 \in E(L)$  and  $w'v_3 \notin E(L)$ .

As  $v_3$  is not simplicial,

$v_3$  has a neighbor  $v'_3 \in S$ .

Now,  $v_2$  and  $v_3$  are of maximum degree, hence by the minimality of  $S$ ,

$$S = \{v_1, v'_3, w'\}.$$

Moreover, we may assume that

$$v'_1 v'_3 \in E(L).$$

(If not,  $\{v'_1, v'_3, v, w'\}$  is a stable cutset in  $L$  disconnecting  $v_1$  and  $w$ .) Hence,

$$wv'_3 \notin E(L),$$

otherwise  $\{v'_1, v'_3, w, v_3\}$  is a forbidden set. Now, as  $A$  is connected,  $d(w) \geq 3$ . We distinguish two subcases.

*Subcase 1.1:  $d(w) = 3$ .*

Let  $w_1$  be the third neighbor of  $w$ . Then  $w_1 \in A$ , and

$$w' \text{ is adjacent to } w_1,$$

otherwise  $\{w, w', w_1, v\}$  is a forbidden set. Hence,

$$d(w') = 3,$$

otherwise every vertex in  $N(w') - \{v_2, w, w_1\}$  together with  $w, w', v_2$  form a forbidden set.

Now, if  $w_1 v'_1 \in E(L)$  then  $w_1 v'_3 \in E(L)$  (otherwise  $\{v'_3, v'_1, w_1, v_1\}$  is forbidden), and  $d(v'_1) = 3$  (every vertex in  $N(v'_1) - \{w_1, v_1, v'_3\}$  together with  $v'_1, w_1, v_1$  form a forbidden set). But then  $A = \{w, w_1, v'_1\}$  and so  $L$  has only nine vertices.

Thus,  $w_1 v'_1 \notin E(L)$ . Then  $w_1 v'_3 \notin E(L)$  (otherwise  $\{v'_3, v'_1, w_1, v_3\}$  is forbidden), and  $\{w_1, v'_3, v_1\}$  is a stable cutset in  $L$  disconnecting  $w$  and  $v'_1$ .

This settles Subcase 1.1.

*Subcase 1.2:  $d(w) = 4$ .* Let  $w_1, w_2$  be the two other neighbors of  $w$ . Then  $w_1, w_2 \in A$ , and as there is no forbidden set,

$$w_1 w_2, w' w_1 \text{ and } w' w_2 \text{ are edges in } L$$

and

$$v'_1 w_1, v'_1 w_2 \text{ and } v'_3 w_1, v'_3 w_2 \text{ are non-edges in } L.$$

As  $w_1$  and  $w_2$  are not simplicial, each of  $w_1$  and  $w_2$  must have a neighbor in  $A - \{w, w_1, w_2\}$ . Let  $w'_1$  be such a neighbor of  $w_1$  and  $w'_2$  be such a neighbor of  $w_2$ . Then

$$w'_1 w_2 \notin E(L) \quad \text{and} \quad w'_2 w_1 \notin E(L)$$

(if say  $w'_1 w_2 \in E(L)$ , i.e.  $w'_1 = w'_2$ , then  $v'_1 w'_1 \notin E(L)$  otherwise  $\{v'_1, w'_1, w_1, w_2, w, w'\}$  is a forbidden set, hence,  $w'_1$  must have a neighbor  $a \in A - \{w_1, w_2, v'_1\}$ . But then  $\{a, w'_1, w_1, w_2, w, w'\}$  is a forbidden set).

If say  $w'_1$  and  $v'_3$  are non-adjacent then  $\{w'_1, w_2, v'_3, v_1\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $v'_1$ . Thus, we may assume that

$$w'_1 \text{ and } v'_3 \text{ are adjacent,} \quad w'_2 \text{ and } v'_3 \text{ are adjacent,}$$

hence,

$v'_1$  and  $w'_1, v'_1$  and  $w'_2$ , and  $w'_1$  and  $w'_2$  are adjacent,

otherwise  $\{v'_3, v'_1, w'_1, v_3\}$ ,  $\{v'_3, v'_1, w'_2, v_3\}$  or  $\{v'_3, w'_1, w'_2, v_3\}$  is a forbidden set.

Now observe that  $\{v, v_1, v_2, v_3, w, w', w_1, w_2, w'_1, w'_2, v'_1, v'_3\}$  induces a 4-regular graph  $G'$  in  $L$ . Hence,  $G' = L$ , a contradiction (because  $L$  has to contain at least 13 vertices). Thus Subcase 1.2, and hence Case 1, is settled.

Case 2:  $w'$  and  $v'_1$  are adjacent.

In this case we first claim

**Claim 2.** Assume  $v_2$  and  $v_3$  have no neighbor in  $S - \{v_1\}$ . Then  $L$  has a stable cutset or

- $v_i$  has exactly one neighbor  $v'_i \in B'$ ,  $i = 2, 3$ ,
- $v'_2$  and  $v'_3$  are adjacent, and
- $N_S(v'_2) = N_S(v'_3)$  and is a set containing one vertex.

(Proof: As  $v_i$  is not simplicial,  $v_i$  has neighbor  $v'_i$  in  $B'$ . By Claim 1,  $v'_2 \neq v'_3$ . If  $v'_2$  and  $v'_3$  are non-adjacent then  $\{v'_2, v'_3, w, v_1\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $v'_1$ . If  $v'_2$  or  $v'_3$  has no neighbor in  $S$ , say  $v'_2$ , then  $(S - \{v_1\}) \cup \{v'_2, v_3\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $v'_3$ . Now, the last statement follows since  $L$  has no induced  $K_{1,3}$ .)

Now, since  $S$  is minimal,  $w'$  has a neighbor in  $B$ . Hence  $A' = A \cap N(w')$  is a clique (otherwise there is an induced  $K_{1,3}$ ), and  $A'$  consists of two or three vertices.

Subcase 2.1:  $A' = \{w, v'_1\}$ .

Then  $d(w) = d(v'_1) = 3$  (otherwise there is an induced  $K_{1,3}$ ). That is,

$$A = \{w, v'_1\}$$

and by the minimality of  $S$ ,

$$S = \{v_1, w'\}.$$

Moreover,  $w'$  cannot be adjacent to both  $v_2$  and  $v_3$  (else  $\{w', v_2, v_3, v, w\}$  is forbidden) and, as  $v_i$  is not simplicial,  $B' \neq \emptyset$ . Hence,  $v_i$  must have a neighbor  $v'_i$  in  $B'$  for some  $i = 2, 3$ , say  $i = 2$ . By Claim 1,  $v'_2$  is non-adjacent to  $v_3$ .

Now, if  $v_3 w' \notin E(L)$  then, by Claim 2, both  $v'_2$  and  $v'_3$  are adjacent to  $w'$ , hence by the minimality of  $S$ ,  $B = \{v_2, v_3, v'_2, v'_3\}$ . But then  $G$  has only nine vertices.

Thus,  $v_3 w' \in E(L)$ . Then  $v'_2 w' \notin E(L)$  (else  $\{w, w', v_3, v'_2\}$  is forbidden), and  $S \cup \{v'_2\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $N(v'_2) - \{v_2\}$  which is non-empty because  $v'_2$  is not simplicial.

Subcase 2.1 is settled.

Subcase 2.2:  $A' = \{w, v'_1, a\}$ .

Let  $b$  be the neighbor of  $w'$  in  $B$ . If  $a$  has a neighbor  $u \in A - \{w, v'_1\}$  then  $\{a\}$  is a stable cutset of  $L$  disconnecting  $u$  and  $v$ . Thus, we may assume that

$$A = \{v'_1, w, a\}.$$

Since  $a$  is not simplicial,  $a$  has (exactly one) neighbor  $a' \in S - \{w'\}$ , and by the minimality of  $S$ ,

$$S = \{v_1, a', w'\}.$$

If  $b \in \{v_2, v_3\}$ , then  $a$  and the vertex in  $\{v_2, v_3\} - \{b\}$  form a stable cutset in  $L$  disconnecting  $v$  and  $a'$ . Thus, we may assume that

$$b \notin \{v_2, v_3\}.$$

Now, if  $a'$  is adjacent to  $b$  then

$$bv_2 \text{ and } bv_3 \text{ are non-edges,}$$

otherwise  $\{b, w', a', v_i\}$  is forbidden for some  $i = 2, 3$ . Hence,

$$a'v_2 \text{ and } a'v_3 \text{ are non-edges,}$$

otherwise  $\{a', a, b, v_i\}$  is forbidden for some  $i = 2, 3$ . Thus, by Claim 2, the neighbor  $v'_2$  of  $v_2$  in  $B'$  is adjacent to the neighbor  $v'_3$  of  $v_3$  in  $B'$ , and  $v'_2, v'_3$  are adjacent to  $a'$ . Then  $b$  must be adjacent to  $v'_2$  and  $v'_3$  (else  $\{a', a, b, v'_i\}$  is forbidden for some  $i = 2, 3$ . Recall that  $b$  is non-adjacent to  $v_2, v_3$ , hence  $b \notin \{v'_2, v'_3\}$ ). But then  $L$  has only 12 vertices.

Thus,  $a'$  is non-adjacent to  $b$ . Then  $\{a', b, w, v_1\}$  is a stable cutset in  $L$  disconnecting  $v$  and  $v'_1$ .

Subcase 2.2, hence Case 2, is settled.

We have shown that, if  $v$  is a bad vertex of maximum degree 4 in  $L$ , then  $L$  has a stable cutset if and only if  $L' = L - v$  has a stable cutset. Note that  $L'$  is again a line graph. Repeat the same reduction for  $L'$  instead of  $L$  if  $L'$  still has a bad vertex of degree 4 and at least 13 vertices. In this way we obtain the desired line graph  $L^*$ .

## 7. Concluding remarks

We have shown that STABLE CUTSET is NP-complete for 5-regular line graphs of bipartite graphs, and that STABLE CUTSET can be solved in polynomial time for line graphs of maximum degree at most 4 and for arbitrary graphs of maximum degree at most 3. Thus, it would be interesting to fill the gap in the complexity status of STABLE CUTSET on arbitrary graphs of maximum degree 4.

**Problem 1.** *Is STABLE CUTSET in P or NP-complete on graphs with maximum degree 4?*

We are going to explain a connection between this problem and another interesting result on stable cutsets. The following nice theorem was conjectured by Caro and has been proved by Chen and Yu [2].

**Theorem 14** (Chen and Yu [2]). *Every graph with  $n$  vertices and at most  $2n - 4$  edges has a stable cutset.*

(This theorem implies that every graph with at least eight vertices and maximum degree at most 3 has a stable cutset, therefore, STABLE CUTSET is in  $\mathbf{P}$  for graphs with maximum degree at most 3; cf. also our discussion in Section 5 without use of Theorem 14.)

Now, let  $G$  be a graph with  $n$  vertices and maximum degree 4. Then  $G$  has at most  $2n$  edges. By Theorem 14, Problem 1 remains open only in four cases, namely for graphs with  $n$  vertices and  $m$  edges where  $2n - 3 \leq m \leq 2n$ .

Thus, the following problem is of interest.

**Problem 2** ( $\text{SCS}(n, m)$ ). *Given a graph  $G$  with  $n$  vertices and at most  $m$  edges. Does  $G$  have a stable cutset?*

Theorem 14 shows that  $\text{SCS}(n, 2n - 4)$  is in  $\mathbf{P}$ , while our Theorem 7 implies that, even on line graphs,  $\text{SCS}(n, (2 + \varepsilon)n)$  is  $\mathbf{NP}$ -complete for every fixed  $\varepsilon > 0$ . A question arises: What is the maximum number  $m$  (depending on  $n$ ) for which  $\text{SCS}(n, m)$  can be solved efficiently? In particular, is  $\text{SCS}(n, 2n - 3)$  in  $\mathbf{P}$ ?

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