UNIVERSITÄT HAMBURG

Mathematisches Seminar

HAMBURGER BEITRÄGE
ZUR MATHEMATIK
aus dem Mathematischen Seminar

Heft 19

Facet Graphs of Pure Simplicial Complexes and Related Concepts

by V. B. Le and E. Prisner

Bundesstraße 55, D-2000 Hamburg 13

Tel.: 040/41 23-51 53

HAMBURGER BEITRÄGE
ZUR MATHEMATIK
aus dem Mathematischen Seminar

Heft 19

Facet Graphs of Pure Simplicial Complexes and Related Concepts

by V. B. Le and E. Prisner

Arbeitsbereich II 1992

Facet Graphs of Pure Simplicial Complexes and Related Concepts

VAN BANG LE *
Fachbereich Mathematatik
TU Berlin

ERICH PRISNER †
Mathematisches Seminar
U Hamburg

January 1992

Abstract. The *k-facet graph* $\Psi_k(K)$ of a pure $(k-1)$-dimensional simplicial complex K has all facets as vertices, and two such distinct vertices are adjacent whenever the corresponding facets of the complex have some common $(k-2)$-dimensional face. In this paper k-facet graphs and related concepts are investigated. Two characterizations are given and a first attempt towards a recognition algorithm is made. Some forbidden induced subgraphs for k-facet graphs are derived, and it is shown that connected k-facet graphs do not have more cliques than edges.

1 Introduction

By a *hypergraph* $H = (A, E)$ we mean a vertex set (mostly called point set) A together with a family $E = (S_i / i \in I)$ of subsets S_i of A, called *(hyper-) edges*. It is *simple* if E is a set, i.e. if all hyperedges are distinct. A hypergraph is a very general mathematical structure — many other structures and many real-world situations form hypergraphs in a natural way.

When facing a hypergraph H one might want to describe the mutual position of any two hyperedges. One very natural way is to state for any pair of hyperedges whether they are disjoint or have nonempty intersection. Then we obtain the well-known intersection graph $\Omega(H)$ of H . It has I as vertex set, and two distinct vertices $i \neq j$ are adjacent whenever the corresponding sets S_i and S_j have nonempty intersection. Intersection graphs have been investigated very intensively for various classes of hypergraphs.

In this paper we are going to introduce some other graphs associated with a simple hypergraph H — a new concept extending several notions of the literature. Here we are no longer satisfied if the hyperedges are nondisjoint, but we ask whether their intersection has a certain shape. For any integer $k \geq 1$, a *k-edge* of H is any edge of cardinality k. For $k \geq 2$, the k-edge graph $\Psi_k(H)$ of H has all k-edges of H as vertices, and two distinct vertices are adjacent whenever their intersection is a $(k-1)$-edge of H. So we have two requests: Firstly the hyperedges corresponding to adjacent vertices should be

*Sekretariat MA 8-1, FB3, TU Berlin, Strasse des 17. Juni 135, 1000 Berlin 12, F.R.Germany
†Mathematisches Seminar, Universität Hamburg, Bundesstr. 55, 2000 Hamburg 13, F.R. Germany
Key words: facet graphs, hypergraphs, simplicial complexes, neighborhoods, cliques, duality,
AMS subject classification: 05C65, 05C75, 57Q, 06, 05B35

should be as close together as they could, i.e. differ in only one element. This is the reason why we compare only hyperedges of the same cardinality. Secondly the intersection of the hyperedges (corresponding to adjacent vertices) should be a hyperedge of H . By requesting only the first condition we arrive at the facet graphs defined below. When requesting only the second condition, we could also admit hyperedges of different cardinality to be compared. This yields another problem, posed by Acharya in [3], that is not treated in this paper. An example of a simple hypergraph H and its 3-edge graph is given in Figure 1. 3-edges of H are indicated by curved lines along three vertices, 2-edges by straight lines between the corresponding two vertices.

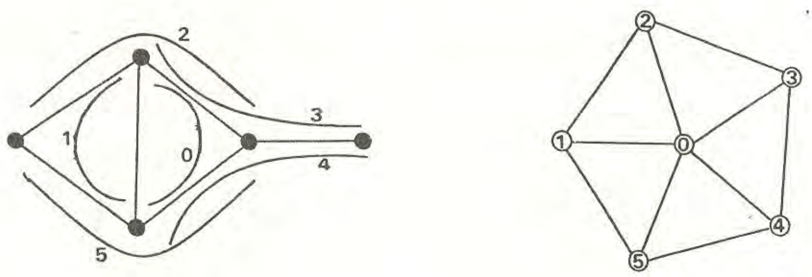

Figure 1
A hypergraph and its 3-edge graph

By restricting the class of simple hypergraphs, we obtain several subclasses of k-edge graphs, that were already known in the literature. The simple hypergraph $K = (A, \Delta)$ is a *simplicial complex*, if $\emptyset \neq S' \subseteq S \in \Delta$ implies $S' \in \Delta$. Edges of K (i.e. elements of Δ) are traditionally called *faces*; the *dimension* of a face is one less than its cardinality. *Facets* are inclusion-maximal faces, and the complex is *pure k-dimensional* if all faces have dimension k. k-edge graphs of pure $(k-1)$-dimensional simplicial complexes K are known under the notation *(k -) facet graph*; we write $F_k(K)$ for this facet graph. Björner asked in [6] which graphs can appear as facet graphs of pure complexes. Our k-facet graphs essentially appeared in [5] and [15] under the name $(k-1)$-line graphs of k-uniform hypergraphs.

The *k-line graph* $L_k(G)$ of a graph G is defined as the k-edge graph of the hypergraph of all *k-simplices* (that are complete subgraphs on k vertices) of G. For $k = 2$ we get the ordinary and famous line graph $L(G) := L_2(G)$, see [14] for a survey on line graphs. More than 20 years ago, several papers appeared on k-line graphs of complete graphs or of complete multipartite graphs, see [4,10] and the literature cited there. k-line graphs play an important role in a characterization of so-called K_i-perfect graphs by Conforti, Corneil, and Mahjoub [8, 9] . In [11], [20], and [24] the iterated behaviour of k-line graphs is investigated. 3-line graphs also appeared in [25] and [11] as triangle graphs. In [28] Tuza posed the problem to characterize 3-line graphs. The set of all complete subgraphs with at most k vertices of a graph is empty or a $(k-1)$-dimensional simplicial complex, thus any k-line graph is a k-facet graph. The hypergraphs appearing in this way are called *conformal.*

Let $P = (P, \leq)$ be a partially ordered set (*poset* for short). A totally ordered subset

$x_o < x_1 < ... < x_t$ is called a *chain* of length t. *Flags* are inclusion-maximal chains. The poset P is *pure of length* k if all flags have the same finite length k. The *(k-) flag graph* $Fl_k(P)$ of a pure poset of length $k-1$ is defined as the k-edge graph of the hypergraph having all chains of P as edges. These graphs have been introduced for special posets (distributive lattices) by Abels [1, 2]. Obviously $Fl_k(P)$ equals the k-line graph of the comparability graph of P, thus any k-flag graph is a k-line graph. 2-flag graphs are just the line graphs of bipartite graphs, see [13].

Beside facet graphs, there is another subclass of k-edge graphs in the literature. A simple finite hypergraph $H = (A, E)$ is a *greedoid* if (1) $\emptyset \in E$, (2) Any k-edge contains some $(k-1)$-edge for any $k \geq 2$, and (3) $X, Y \in E$ and $|X| = |Y|+1$ implies $Y \cup \{x\} \in E$ for some $x \in X \setminus Y$. Inclusion-maximal edges are called *bases*, and (3) implies that all bases have the same cardinality k. Then the *basis graph* of the greedoid [7, 18] is defined as its k-edge graph. A *matroid* is a greedoid that is also a simplicial complex. Basis graphs of matroids were investigated in [16, 17, 22, 23] and characterized in [22].

When facing all this graph classes of k-edge graphs, k-facet graphs, and so on, the most natural question is that for characterizations. However, in general this seems to be difficult. Although we are going to present two characterizations in Sections 4 and 6, they seem not to yield directly efficent recognition algorithms. Thus the challenge to find good characterizations remains, and it is also the purpose of the present paper, to stimulate research in this direction.

In Section 2 we describe the order between the graph classes defined, i.e. which one is contained in another.

Section 3 is mainly devoted to local observations. It can be shown that the neighborhoods or the common neighborhoods of k-edge graphs must fulfil certain properties. From this we derive some forbidden induced subgraphs.

The first characterization for line graphs was given by Krausz in [19]. Grünbaum pointed out in [12] that there is no Krausz-type characterization for facet graphs. However, if we add certain ingredients, extensions of Krausz's characterization are possible. The underlying idea is in both cases a sort of duality principle. In Section 4 we give the first such characterization.

In Section 5 we show that the cliques in facet graphs can be naturally partitioned into two classes — so-called star cliques and hole cliques. The mutual position of these cliques can be described. In particular facet graphs do not have too many cliques.

In Section 6 we give our second Krausz-type characterization. Using the results of Section 5 we construct a poynomial-time algorithm that reduces the problem to recognize facet graphs to another one, recognizing a sort of hypergraphs.

Throughout this paper, by a *complex* we mean always a simplicial complex. All graphs here are simple, that is, without multiple edges and loops; but not necessarily finite. For a subset W of the vertex set of the graph G, $G[W]$ denotes the subgraph of G induced by W. Graphs that do not contain an induced subgraph isomorphic to a given graph F are called *F-free.* The *join* $G * H$ of two graphs G and H is obtained from the disjoint union of G and H by adding all edges between vertices of G and vertices of H. For a vertex x of G we denote by $N_G(x)$ the *neighbourhood* of x in G, that is, the set of all vertices of G adjacent to x; when there can be no confusion we shall write $N(x)$ for $N_G(x)$. By $K_n, K_n - e, K_{m,n}, C_n, W_n$, and P_n we mean the complete graph on n vertices, the complete graph on n vertices with one edge deleted, the complete bipartite graph with the bipartition on m and n vertices, the cycle on n vertices and n edges, the wheel $K_1 * C_n$, and the path on n vertices and $n-1$ edges, respectively. Finally, a complete subgraph (on t vertices) of a graph is also called (t-)*simplex* of the graph, and a *clique (t-clique)* is a inclusion-maximal simplex (t-simplex).

In order to avoid irritations, we strictly separate between the hypergraph $H = (A, E)$ and its k-edge graph $G = (V, E)$. We call the elements of H *points*, and denote them by $a, b, c, \ldots$. The edges of H are denoted by $X, Y, Z, \ldots$, and the corresponding vertices of G by $x, y, z, \ldots$.

2 Relation between the graph classes

The relation between the classes for fixed $k \geq 2$ is sketched in Figure 2.

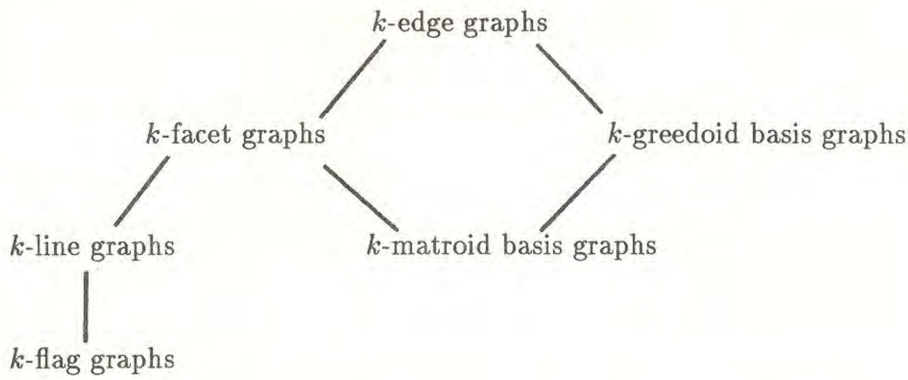

Figure 2
Relation between our graph classes

The mutual position of the different graph classes (for fixed k) was already sketched in the introduction. In order to compare corresponding classes for different k, we introduce another definition. For any hypergraph H , by (a, H) we mean the hypergraph with one additional new point a and all edges $S_i \cup \{a\}$, where $S_i \in E(H)$. If H is a simplicial complex, then (a, H) denotes the cone. Obviously $\Psi_k(H) \simeq \Psi_{k+1}((a, H))$. Thus, if a family $\mathbf{H}$ of simple hypergraphs is closed under this cone-operation, then any k-edge graph of members in $\mathbf{H}$ is also $(k+1)$-edge graphs of members in $\mathbf{H}$. This holds for all our hypergraph classes considered except in the k-line graph case. So any k-edge graph is a $(k+1)$-edge graph, any k-facet graph is a $(k+1)$-facet graph, and so on. Thus we get sequences of graph classes, for instance 2-facet graphs, 3-facet graphs, 4-facet graphs, ... each one including the previous one. Thus by *edge graphs, facet graphs, flag graphs, greedoid basis graphs*, and *matroid basis graphs*, we denote the graphs in the unions of the corresponding inclusion-sequences of sets.

Now the most general class of graphs considered in this paper is the class of edge graphs. This class is fairly general, since we can show:

Proposition 2.1 *Every finite graph $G = (V, E)$ is an edge graph.*

Proof: We show that G is a $(|V| - 1)$-edge graph by constructing a representation, i.e. a simple hypergraph H. As point set of H we choose again V. Then we sample all possible $(|V|-1)$-edges in H. For $M \subseteq V$ with $|M| = |V| - 2$, we take M as edge in H if and only if the two vertices in $V \setminus M$ are adjacent in G. It is easy to see that $\Psi_{|V|-1}(H) \simeq G$ by the bijection $x \to V \setminus \{x\}$ from V into the set of $(|V|-1)$-edges of H. □

Note also that any graph is an intersection graph [21].

Next we investigate the relation of the graph classes for fixed small k.

Proposition 2.2 *2-edge graphs, 2-facet graphs, (2-) line graphs mean all the same.*

Proof: We show that any 2-edge graph is a 2-line graph. Let $G = \Psi_2(H)$. Let M denote the set of those points b in H for which $\{b\}$ forms an edge in H, and let N denote the set of all other points. We replace each $b \in N$ by new points b_e for all 2-edges e of H containing b. Now we take $M \cup \{b_e / b \in N, b \in e, |e| = 2, e \in E(H)\}$ as vertex set of a graph F. For any 2-edge $e = \{a, b\}$ of H we add an edge between a, b or a, b_e or a_e, b_e or a_e, b, depending on $a, b \in M$ or N. Then $L_2(F) = \Psi_2(H)$. □.

2-flag graphs, 2-greedoid basis graphs, and 2-matroid basis graphs are three distinct classes. 2-flag graphs are just the well-known line graphs of bipartite graphs, see [13]. It can be shown that the diameter of any 2-greedoid basis graph is at most 2, and that any two distance 2 vertices in a 2-matroid basis graph must even have 2 common neighbors.

For $k \geq 3$, we can separate k-edge graphs, k-facet graphs, k-line graphs, and k-flag graphs using results that we shall prove later. The wheel W_5 is a 3-edge graph (see Figure 1) and thus a k-edge graph. However it is no facet graph (see Corollary 3.3). The graph $K_4 - e$ is a 3-facet graph (and thus a k-facet graph for any $k \geq 3$) but no k-line graph for $k \geq 3$ (see Section 5). The graph $K_{k+3} - K_{2,k-1}$ is a k-line graph (see Figure 3), since $K_{k+3} - K_{2,k-1} \simeq L_k(K_{k+3} - (K_4 - e))$, but none of these graphs is a flag graph (see again Section 5).

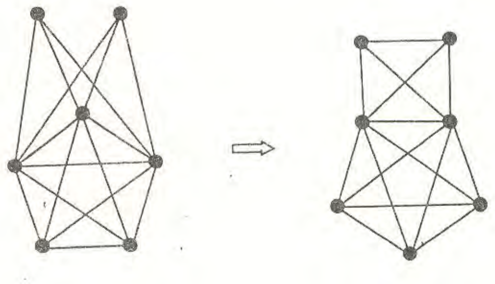

Figure 3
A graph and its 4-line graph

3 Neighborhoods and obstructions

A class of graphs is *hereditary* (or *closed under induced subgraphs*) if any induced subgraph of any member of the class lies in the class also. Such hereditary classes can always be characterized by a (possibly infinite) list of forbidden induced subgraphs.

Proposition 3.1 *For every $k \geq 2$, each one of the class of k-edge graphs and k-facet graphs is hereditary.* □

In the same way, edge graphs and facet graphs are hereditary. A (incomplete) list of forbidden induced subgraphs for facet graphs has been given in [15,Figures 3 and 4]. This is different in the k-line graph case. For instance, we shall see in Section 5 that the graph $K_4 - e$ is no k-line graph for any $k \geq 3$, while it is an induced subgraph of the k-line graphs $K_{k-1} * \overline{P_3}$ for any $k \geq 2$. (Maybe this indicates that characterizing k-line graphs is more difficult than characterizing k-facet graphs.)

We now discuss the local structure of k-edge graphs. When looking at neighborhoods in a k-edge graph, one could also represent the vertices by edges of a bipartite graph rather than by edges of a simple hypergraph. This has the advantage that edges are quite easier to handle than k-edges.

Theorem 3.2 *Let x be a vertex of the k-edge graph G, where $k \geq 2$. Then there is some bijection $y \to e_y$ between $N_G(x)$ and the edges of some bipartite graph B with vertex partition $W \cup X$ with the following properties:*

(1) $|X| = k$,

(2) The edges e_y, e_z corresponding to adjacent vertices $y, z \in N_G(x)$ have nonempty intersection.

(3) For all $y \neq z \in N_G(x)$, $e_y \cap e_z \in X$ implies $yz \in E(G)$.

Proof: Let $G = \Psi_k(H)$ for some simple hypergraph $H = (A, E)$, and let X denote the k-edge of H corresponding to x. We define $W := A \setminus X$. Next we define the edges of B by defining the bijection $y \to e_y$. Let y be some neighbor of x in G, and let Y denote the corresponding k-edge of H. $X \setminus Y$ contains just one element, say a, and $Y \setminus X$ contains just one element, say b $(\in W)$. We define $e_y := ab$. (1) is obvious, we claim that (2) and (3) hold.

To see (2), let us assume $e_y = ab$ and $e_z = cd$ for $y \neq z \in N_G(x)$. If ab and cd are disjoint edges, then $|Y \cap Z| \leq k - 2$. Then y and z are not adjacent.

For (3), assume $e_y \cap e_z = \{a\}, a \in X$, for $y \neq z \in N_G(x)$. For the corresponding k-edges Y and Z this implies $Y \cap Z = X \setminus \{a\}$, i. e. $Y \cap Z = Y \cap X$. But $Y \cap X$ is an $(k-1)$-edge in H, since $y \in N_G(x)$, whence $yz \in E(G)$. □.

When we concentrate on k-facet graphs, it is easy to see that we could request that $e_y \cap e_z \neq \emptyset$ implies $yz \in E(G)$ instead of (3). This means nothing than the line graph of B is isomorphic to the subgraph of G induced by $N_G(x)$.

Corollary 3.3 ([15]) *Every neighborhood in a k-facet graph G induces the line graph of some bipartite graph with one partition class having at most k elements.* □

Moreover it could be shown that every line graph of a bipartite graph apears as neighborhood in some facet graph.

Corollary 3.4 ([15]) *Every facet graph is $(K_5 - e)$-free, $(K_2 * \overline{K_3})$-free, and W_{2t+1}-free for all integers $t \geq 2$, see Figure 4.*

Proof: In [13] it has been shown that line graphs of bipartite graphs are $(K_4 - e)$-free, $K_{1,3}$-free, and C_{2t+1}-free, for all integers $t \geq 2$. □.

By the remark succeding Corollary 3.3 there is no other forbidden induced subgraph with one vertex adjacent to all others, except, of course, supergraphs of the graphs in Figure 4.

No equivalent of Corollary 3.4 is possible for general edge graphs, see Proposition 2.1. However, for small k, Theorem 3.2 could as well be useful. First we reformulate it slightly, first two more definitions: We call an edge *unicliqual* if it lies in only one clique. An *isolated* clique is one containing only uniciqual edges.

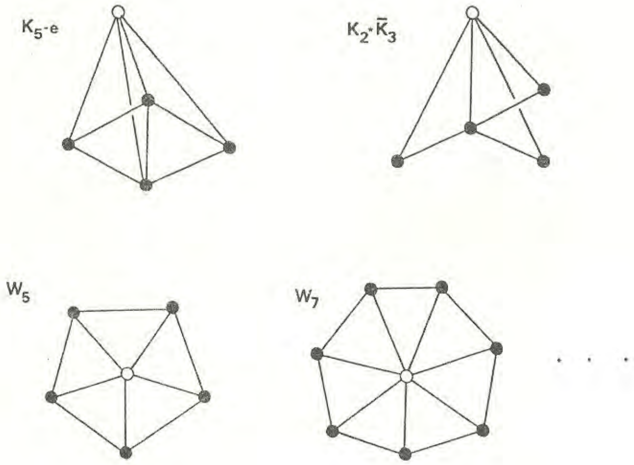

Figure 4
Some forbidden induced subgraphs for facet graphs

Corollary 3.5 *Let G be a k-edge graph, with $k \geq 2$. Then every neighborhood $N_G(x)$, $x \in V(G)$, has some partition by at most k isolated cliques and 1-simplices of $G[N_G(x)]$.*

Proof: Let $y \to e_y$ be the bijection between $N_G(x)$ and the edge set of some bipartite graph B with bipartition $W \cup X$ as in Theorem 3.2, and let $X = \{a_1, a_2, \ldots, a_k\}$. We define $Q_i := \{y \in N_G(x)/a_i \in e_y\}$, for $i = 1, 2, \ldots, k$. Obviously $Q_1, Q_2, \ldots, Q_k$ is a partition of $N_G(x)$. According to condition (3) in Theorem 3.2, each such Q_i induces a complete subgraph in G (it may be empty).

Assume now $|Q_i| \geq 2$.

We claim that no vertex of $N_G(x)$ outside Q_i is adjacent to at least two vertices in Q_i. Assume on the contrary there were $y_1, y_2 \in Q_i, z \notin Q_i$ with $zy_1, zy_2 \in E$. By condition (2) of Theorem 3.2, e_z and e_{y_1}, as well as e_z and e_{y_2} must have common endpoints, but clearly not in X. But the endpoints of e_{y_1} and e_{y_2} in W are distinct, a contradiction.

Now such a Q_i, containing at least 2 vertices, must be a clique by the claim above. By the same reason, it must be an isolated clique, and we are done. $\square$.

From this we can also derive forbidden induced subgraphs for k-edge graphs and fixed k. Let us give some examples:

Corollary 3.6 *For every $k \geq 2$, every k-edge graph is $K_{1,k+1}$-free and W_{2k+1}-free.* $\square$

Corollary 3.7 *For every $k \geq 3$, every k-edge graph is $(K_{k+2} - e)$-free.*

Proof: Assume $G = \Psi_k(H)$ contains a $(K_{k+2} - e)$, then some neighborhood $N_G(x)$ contains some $F \simeq K_{k+1} - e$. It is easy to see that no isolated clique can meet F in more than one vertex, provided $k \geq 3$. But F has $k+1$ vertices, and we get a contradiction to Corollary 3.5. $\square$.

Next we extend an idea of Maurer [22] and Holzmann, Norton, Tobey [17]. We look at the common neighbourhood $N(x) \cap N(y)$ of two nonadjacent vertices x, y, and derive a sort of common neighborhood condition.

Theorem 3.8 *Let x and y be to nonadjacent vertices of the facet graph G. Then $G[N(x) \cap N(y)]$ is an induced subgraph of C_4.*

Proof: Let $G = F_k(K)$ be the k-facet graph of the $(k-1)$-dimensional pure simplicial complex K, and let x, y be two nonadjacent vertices of G. Let X and Y denote facets of K corresponding to x and y. $|X \cap Y| = k-2$, then say $X = \{a_1, a_2, c_1, \ldots, c_{k-2}\}$ and $Y = \{b_1, b_2, c_1, \ldots, c_{k-2}\}$. Then there are four possible common neighbors of x and y, whose k-edges correspond to $\{a_i, b_j, c_1, \ldots, c_{k-2}\}$ with $i, j \in \{1, 2\}$. $\square$

Again, by Proposition 2.1, no such result is possible for edge graphs — distance 2 vertices may correspond to k-edges of H whose intersection has cardinality $k-1$.

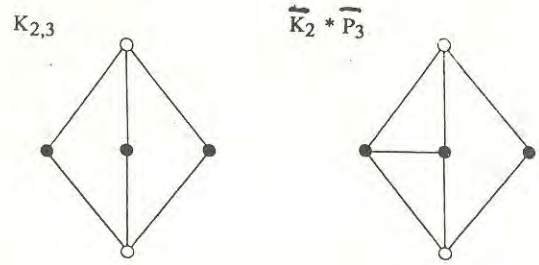

Figure 5
More forbidden induced subgraphs for facet graphs

Corollary 3.9 ([15]) *Any facet graph is $K_{2,3}$-free and $\overline{K_2} * \overline{P_3}$-free (and also $(K_5 - e)$-free as has been shown already), see Figure 5.* $\square$

More forbidden induced subgraphs for facet graphs can be found in [15].

4 A first characterization

Let $G = \Psi_k(H)$ be the k-edge graph of a simple hypergraph $H = (A, E)$. For each $a \in A$, let S_a be the set of all k-edges of H that contain the point a. Note that for every subset $A' \subseteq A$ and every k-edge X of H,

$$X \in \bigcup_{a \in A'} S_a \text{ if and only if } A' \subseteq X,$$

$$\text{moreover, } \{X\} = \bigcap_{a \in A'} S_a \text{ if and only if } A' = X.$$

Now let G_a be the subgraph of G induced by the vertices corresponding to members of S_a, and define $\Theta_k = (G_a / a \in A)$, the family of induced subgraphs G_a of G. Then Θ_k has the following properties:

(1) Any edge of G is contained in exactly $k-1$ members of Θ_k .

Proof of (1). Let xy be an edge of G, and let X, Y be the k-edges corresponding to the vertices x and y, respectively. Then $|X \cap Y| = k-1$, and just the $k-1$ sets $S_a, a \in X \cap Y$, contain both X and Y. That is, exactly the $G_a, a \in X \cap Y$ contain both x and y . Since the G_a are induced in G, the edge xy is contained in exactly these $k-1$ members of Θ_k. □

(2) Any vertex of G is contained in exactly k members of Θ_k .

Proof of (2). Let X be the k-edge corresponding to a vertex x of G. Then the intersection of all $S_a, a \in X$, consists of exactly X; that is $\{x\} = \bigcap_{a \in X} V(G_a)$. □

(3) $|\bigcap_{a \in A'} V(G_a)| \leq 1$ for any subset A' of order k of A.

Proof of (3). Let X and Y be elements of the intersection of all $S_a, a \in A'$. Then $A' \subseteq X, A' \subseteq Y$. Thus if A' is of order k, then $A' = X = Y$. □

(4) $\bigcap_{a \in A'} G_a$ is complete or edgeless, for any subset A' of order $k-1$ of A.

If H is moreover a complex, then instead of (4) we have the following stronger property:

(5) $\bigcap_{a \in A'} G_a$ is a complete subgraph of G, for any subset A' of order $k-1$ of A.

Proof of (4) and (5). Let X and Y be distinct elements of the intersection of all $S_a, a \in A'$. Then $X \cap Y \supseteq A'$; thus the corresponding vertices x and y are adjacent, if A' is a $(k-1)$-edge, in the hypergraph case; in the case of a complex H , x and y are adjacent if A' is of order $k-1$. □

We now consider the graph case, where $J = (A, E)$ is a graph and $G = L_k(J)$, the k-line graph of J. Then G is also a k-facet graph, as noted in the introduction. Now, the sets S_a , defined at the beginning of this section, consist just of all k-simplices of J which contain the vertex $a \in A$ of J. The family Θ_k has, beside the four properties (1), (2), (3) and (5), the following fifth:

(6) Θ_k has the k-Helly property.

That means, for any k-element subset A' of A, if $G_a, a \in A'$, are pairwise intersecting, then the intersection of all these $G_a, a \in A'$, is nonempty.

Proof of (6). Let $x \in V(G_{a_1} \cap G_{a_2})$ and let X be the k-simplex corresponding to the vertex x. Then $X \in S_{a_1} \cap S_{a_2}$; that is, a_1 and a_2 are vertices of X, therefore a_1 and a_2 are adjacent in J. Thus, if $S_a, a \in A'$, are pairwise intersecting, then the vertices in A' are pairwise adjacent. If A' is of order k, then so the k-simplex induced by A' is the intersection of all the $S_a, a \in A'$. □

Theorem 4.1 *(A first characterization) Let $k \geq 2$ be an integer. A graph G is*
(i) the k-edge graph of some simple hypergraph,
(ii) the k-facet graph of some pure $(k-1)$-dimensional simplicial complex,
(iii) the k-line graph of some graph,

if and only if there is some family $\Theta_k = (G_a / a \in A)$ of induced subgraphs of G which obeys
(i) properties (1) - (4) above,
(ii) properties (1), (2), (3), and (5),
(iii) properties (1), (2), (3), (5), and (6).

Proof: Because of the discussion above, all we have to show is the 'only if'- part of the theorem. Let $\Theta_k = (G_a / a \in A)$ be some family of induced subgraphs of G obeying (1), (2), and (3) above. We call a k-element subset $\{a_1, \ldots, a_k\}$ of A a *good* set, if $\bigcap_{i=1}^{k} G_{a_i}$ is nonempty, say $\{y\} = \bigcap_{i=1}^{k} V(G_{a_i})$ by (3). We consider the simple hypergraph H on the point set A with all good k-subsets of A as its k-edges, and where $\{a_1, \ldots, a_{k-1}\}$ is a $(k-1)$-edge of H if and only if $\bigcap_{i=1}^{k-1} G_{a_i}$ is complete and nonempty. Then we define $\varphi : \Psi_k(H) \to G$ as follows: Let x be any vertex of $\Psi_k(H)$, and let $X = \{a_1, \ldots, a_k\}$ denote the corresponding k-edge in H, a good subset of A. Now we choose $\varphi(x) := y$, where $\{y\} = \bigcap_{i=1}^{k} V(G_{a_i})$. By (3), φ is well-defined, and by (2), it is bijective.

(i) Assume that Θ_k also fulfills Property (4). We claim that φ is a graph-isomorphism.

Let xx' be any edge of $\Psi_k(H)$, where the k-edges $X = \{a_1, \ldots, a_{k-1}, b\}$ and $X' = \{a_1, \ldots, a_{k-1}, b'\}$ of H correspond to the vertices x and x' of $\Psi_k(H)$, respectively. By the definition of the k-edge graph, $\{a_1, \ldots, a_{k-1}\}$ is a $(k-1)$-edge of H, so $S := \bigcap_{i=1}^{k-1} G_{a_i}$ is complete. Since $\{\varphi(x)\} = V(S \cap G_b)$ and $\{\varphi(x')\} = V(S \cap G_{b'})$, S certainly contains both $\varphi(x)$ and $\varphi(x')$, whence $\varphi(x)\varphi(x') \in E(G)$.

Conversely, let yy' be any edge of G. By (1), there are $a_1, a_2, \ldots, a_{k-1} \in A$ such that yy' lies in $S := \bigcap_{i=1}^{k-1} G_{a_i}$. By (2) and (3), there are $b \neq b' \in A \setminus \{a_1, \ldots, a_{k-1}\}$ such that $\{y\} = V(S \cap G_b)$ and $\{y'\} = V(S \cap G_{b'})$. Thus, by definition, $X = \{a_1, \ldots, a_{k-1}, b\}$ and $X' = \{a_1, \ldots, a_{k-1}, b'\}$ are good subsets of A, that is, X and X' are k-edges of H. By the definition of φ, $y = \varphi(x)$ and $y' = \varphi(x')$, where x, x' are the vertices of $\Psi_k(H)$ that correspond to X, X', respectively. Now S contains the edge yy', hence S is nonempty, and by (4) complete. Consequently $\{a_1, \ldots, a_{k-1}\} = X \cap X'$ is a $(k-1)$-edge of H, therefore x and x' are adjacent in $\Psi_k(H)$.

(ii) Next assume that instead of (4) the stronger property (5) holds. Then, by the construction of the hypergraph H, a subset $\{a_1, \ldots, a_{k-1}\}$ of H forms an edge if and only if $\bigcap_{i=1}^{k-1} G_{a_i}$ is nonempty. This is certainly true for all $(k-1)$-element subsets of good sets in A. This means, that we can add edges of cardinalities smaller than $k-1$ in order to obtain a simplicial complex H' with the same k-edge graph as H. Thus G is even a k-facet graph in this case.

(iii) Finally assume that Θ_k obeys furthermore properties (5) and (6). We have already shown in (ii) that there is some hypergraph H with $G \simeq \Psi_k(H)$. Now look at the underlying graph $J = (A, E_J)$ of H. We claim $G \simeq L_k(J)$ (defined by $ab \in E_J$ if there is some hyperedge containing both a and b). All what remains to show is that any k-simplex $\{a_1, \ldots, a_k\}$ of J lies already in H, since we have already seen in (ii) that the additional (5) implies that the $(k-1)$-edges of H are all $(k-1)$-element subset of the k-edges. So assume that for every pair $i \neq j \in \{1, \ldots, k\}$ a_i and a_j belong to some common k-edge of H, that is, in particular, $G_{a_i} \cap G_{a_j} \neq \emptyset$. By property (6), $\bigcap_{i=1}^{k} G_{a_i}$ is nonempty, and thus $\{a_1, a_2, \ldots, a_k\}$ is a k-edge of H.

□.

Note that for $k = 2$, Theorem 4.1 is just Krausz's characterization of line graphs.

When applying this characterization to the example in Figure 1, we obtain the partition Θ_k of G as in Figure 6.

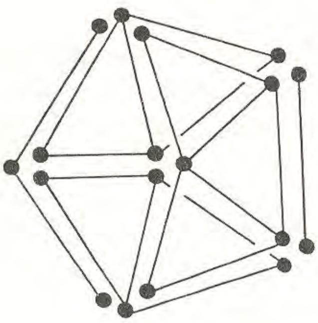

Figure 6
Some Θ_k obeying (1) - (6) of Theorem 4.1

Let us also remark that it is not too difficult to show that all these induced subgraphs G_a must be $(k-1)$-edge graphs, or $(k-1)$-facet graphs, or $(k-1)$-line graphs, respectively.

5 Cliques in facet graphs

Let us take a closer look on the cliques of the k-facet graph G of a complex K . Let us assume for simplicity that G has no isolated vertices. Let $C = \{x_1, \ldots, x_t\}$ be some clique in G. There are two possibilities how the facets $X_1, \ldots, X_t$ corresponding to $x_1, \ldots, x_t$ are located in K. First they could all share some common $(k-1)$-face, i. e. $|\bigcap X_i| = k - 1$. In this case we call C a *star clique.* Otherwise it is easy to see that $|\bigcup X_i| = k + 1$, and $X_1, \ldots, X_t$ are all the k-element subsets of $\bigcup X_i$ which lie in K. In this second case we call C a *hole clique*, since $\bigcup X_i$ forms something like a hole' in K. Note that in the k-edge graph case we could only say that $X_1, \ldots, X_t$ are **some** such subsets, whence this approach is not approbriate for k-edge graphs. Note also that in the k-line graph case all hole cliques have the same cardinality $t = k + 1$.

Clearly the partition of the cliques of G into star cliques and hole cliques depends on the representation of G as facet graph. It will be shown that finding this partition is possible even when the representation is not given.

2-cliques are star cliques (they also obey the properties for hole cliques, but we have chosen the definition exclusive). 1-cliques do not fit into this scheme, and this is the reason why we excluded isolated vertices in G. Any m-clique with $m \geq k + 2$ must be a star clique, since any $(k + 1)$-element set has only $k + 1$ subsets of cardinality k. In the k-line graph case all m-cliques with $m \neq k + 1$ must be star cliques.

We need some more definitions: The *clique graph* $C(G)$ of a graph is defined as the intersection graph of the set of all cliques of G. In other words, $C(G)$ has all cliques of

G as vertices, and two such vertices are adjacent whenever the cliques have nonempty intersection. The *weighted clique graph* $C_w(G)$ of a graph G is the clique graph with edge weights — every edge CD is weighted by $w(CD) := |V(C \cap D)|$. When we draw such a weighted clique graph, edges weighted by the natural number j are indicated by j parallel edges, see Figure 9 for an example. All edges of weight 2 generate a subgraph of $C(G)$ (with the same vertex set as $C(G)$), called the *2-overlap clique graph* of G in [26] and denoted by $g_2(G)$.

Theorem 5.1 *Let G be a k-facet graph for some integer $k \geq 2$. Then all edge-weights of $C_w(G)$ are 1 or 2. Moreover, there is some partition $\kappa_S \cup \kappa_H$ of the vertex set κ of $C_w(G)$ such that all edges of weight 2 join vertices of different classes, and every weight-1 edge joins vertices of the same class. Every vertex of κ_H corresponds to some j-clique of G, with $3 \leq j \leq k+1$. If G is even a k-line graph, then all vertices of κ_H correspond to $(k+1)$-cliques of G.*

Proof: If there were some edge weight greater than 2, then we would have an induced subgraph isomorphic to $K_5 - e$ in G, a contradiction to Corollary 3.4. We define κ_H as the set of hole cliques, and κ_S as the set of star cliques of G. It is easy to see that no two star cliques have a common edge, and no two hole cliques have a common edge (this latter were not true in the mere k-edge case). Moreover it is also straightforward to show that if a star clique and a hole clique have nonempty intersection, then they have exactly two common vertices. The rest follows with the remarks above. □.

Corollary 5.2 *The 2-overlap clique graph $g_2(G)$ of every facet graph G is bipartite.* □

Now we are able to demonstrate the independence of the classes of 2-line graphs, 3-line graphs, and so on. We present a graph sequence $G_2, G_3, \ldots$, where each graph G_i is k-line graph only for $k = i$. We choose $G_i := L_i(K_{i-1} * (\overline{K_{i-1}} \cup K_2))$, that is, G_i consists of two $(i+1)$-cliques with one common edge. Clearly $G_2 \simeq K_4 - e$.

Proposition 5.3 *For all integers $i, k \geq 2$, the graph G_i is a k-line graph if and only if $k = i$.*

Proof: Assume G_i is a k-line graph. Surely $C(G_i)$ is isomorphic to K_2, and the edge is weighted by 2. So, in any case, κ_H contains one vertex of $C(G_i)$ corresponding to an $(i+1)$-clique of G_i. By Theorem 5.1, $k = i$. □

k-line graphs of K_{k+1}-free graphs J have even a simpler structure. The k-line graph $G = L_k(J)$ can not contain hole cliques. In particular G must be $(K_4 - e)$-free, and $g_2(G)$ edgeless. In particular, (k-) flag graphs belong to this class. Maybe a characterization of (k-) flag graphs is not too difficult.

Facet graphs — even $(K_5 - e)$-free graphs — have few cliques in the following sense (this Theorem 5.4 is needed in Algorithm 6.7):

Theorem 5.4 *Every finite $(K_5 - e)$-free graph $G = (V, E)$ contains at most $\frac{|V|^3}{4}$ cliques.*

Proof: First note that any $(K_4 - e)$-free graph with n_0 isolated vertices and m edges contains at most $n_0 + m$ cliques, since each edge lies in exactly one clique then.

We may assume w. l. o. g. that the $(K_5 - e)$-free graph G contains no isolated vertices. Then we get all cliques of G by looking at all neighborhoods $N(x)$ and at all cliques of these graphs $G[N(x)]$. In fact, we get any k-clique exactly k times, any each clique appears at least twice. Each graph $G[N(x)]$ is $(K_4 - e)$-free. By the remarks above we get at most $|V|\frac{|V|^2}{2}$ entries in the list. Since each clique appears at least twice, we get the result. □.

For facet graphs we can even give a much better bound:

Theorem 5.5 *Finite facet graphs $G = (V, E)$ without isolated vertices have at most $|E|$ cliques.*

Proof: Let E_0 the set of isolated edges (edges that lie in no triangle) in the facet graph G. We have shown above that the set of all hole cliques forms an edge-disjoint cover of the edges of $G - E_0$. The set of all star cliques of cardinality at least 3 forms also such a cover. Since the cardinality of any of these cliques is at least 3, there are at most $\frac{|E \setminus E_0|}{3}$ hole cliques and at most $\frac{|E \setminus E_0|}{3}$ star cliques of cardinality at least 3 also. The remaining cliques are the 2-cliques, their number is $|E_0|$. Thus there are at most

$$\frac{2}{3}|E \setminus E_0| + |E_0| \leq |E|$$

cliques in G. □.

Applying an algorithm in [27] there follows:

Corollary 5.6 *All cliques of a finite facet graph $G = (V, E)$ can be listed in time $O(|V||E|^2)$.* □

Corollary 5.7 *A maximum clique can be found in time $O(|V||E|^2)$ in finite facet graphs.* □

6 A second characterization

Our first characterization in Section 4 works with the dual H_k^* of the hypergraph H_k that contains exactly the k-edges of H. In this section we give another characterization that uses the dual of some other hypergraph derived from H. Let $k \geq 2$ be a fixed integer, and let H be a simple hypergraph. The *$(k-1,k)$-hypergraph* $\Phi_{k-1,k}(H)$ of H has all $(k-1)$-edges of H as vertices. For any k-edge Y of H we derive an edge Y' in $\Phi_{k-1,k}(H)$ by

$$Y' := \{S \subseteq Y / S \text{ is an } (k-1)-\text{edge of } H\}.$$

$\Phi_{k-1,k}(H)$ is not necessarily simple, but if H is a complex, then it is. Now two such edges Y', Z' contain some common element (have nonempty intersection) if and only if the corresponding k-edges Y, Z in H contain both some common $(k-1)$-edge. This latter is equivalent to $|Y \cap Z| \in E(H)$ for $Y \neq Z$.

Proposition 6.1 $\Omega(\Phi_{k-1,k}(H)) \simeq \Psi_k(H)$ *for any simple hypergraph H.* □

How duality is involved can be better seen in the reformulation that $\Psi_k(H)$ is isomorphic to the underlying graph of the dual $(\Phi_{k-1,k}(H))^*$. Another reformulation is $(\Phi_{k-1,k}(H))^*$ forms a simplex-edge cover of $\Psi_k(H)$: Each hyperedge in $(\Phi_{k-1,k}(H))^*$ induces a simplex in $\Psi_k(H)$, and each edge of $\Psi_k(H)$ lies in some hyperedge of $(\Phi_{k-1,k}(H))^*$.

Theorem 6.2 *Let* **H** *be any class of simple hypergraphs, and let $k \geq 2$ be an integer. Then a graph G is the k-edge graph of some member of* **H** *if and only if there is some simplex-edge cover S of G that is the dual of the $(k-1,k)$-hypergraph of some member of* **H**. □

This characterization is not good (in the sense of Edmonds) for two reasons: Firstly we have no good characterization of $(k-1,k)$-hypergraphs (of members of certain classes). Secondly there is an exponential number of simplex-edge covers of a graphs — we can not check all. Both problems are attacked in the following two subsections, however only the second is solved completely.

6.1 $(k-1,k)$-hypergraphs

Proposition 6.3 *Let $k \geq 2$ be an integer and H a simple hypergraph. Then every edge of $\Phi_{k-1,k}(H)$ has cardinality at most k . Moreover no two distinct edges of $\Phi_{k-1,k}(H)$ contain more than one common element. If H is a complex, then $\Phi_{k-1,k}(H)$ is a simple k-uniform hypergraph.*

Proof: The first statement follows from the fact that any k-edge Y of H contains at most k $(k-1)$-edges. Any two such $(k-1)$-edges uniquely determine Y, whence the second result. The alloy for complexes is easy to see. □.

However this two properties are not sufficient, as can be seen by the hypergraph given in Figure 7. It can be shown that this hypergraph is no (2,3)-hypergraph of any simple hypergraph.

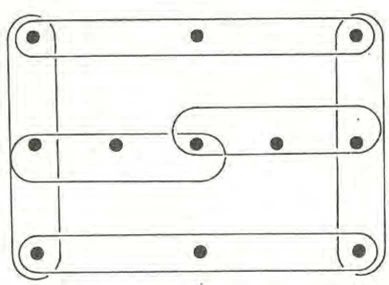

Figure 7
A hypergraph, but no (2,3)-hypergraph

Problem 6.4 *Which hypergraphs are $(k-1,k)$-hypergraphs of simple hypergraphs, or simplicial complexes, or conformal simplicial complexes?*

A solution of this problem would acquit us of all our sorrow, as is shown in Subsection 6.2.

Next we investigate a very special class of hypergraphs for being $(k-1,k)$-hypergraph. A *block graph* is a graph where any 2-connected induced subgraph is complete. A hypergraph $H=(A,E)$ is called *Berge-acyclic*, if there are no $j \geq 2$ and no sets $\{a_1,\ldots,a_j\}$

of points and $\{X_1, \ldots, X_j\}$ of hyperedges with $a_i \in X_i \cap X_{i+1}$ for $i = 1, \ldots, j$ and the indices modulo j. Clearly no two edges of a Berge-acyclic hypergraph have two (or more) common points. It is well-known that a hypergraph is Berge-acyclic if and only if it is conformal, and its underlying graph is a block graph.

Recall that the *chain complex* $\Delta(P)$ of a poset P has all chains as simplices.

Proposition 6.5 *For every $k \geq 2$, every finite k-uniform Berge-acyclic hypergraph is the $(k-1,k)$-hypergraph of the chain complex of some poset of length k.*

Proof: We choose induction over the edge number of the hypergraph. Let $H = (V, E)$ be some finite k-uniform Berge-acyclic hypergraph, and let the statement be true for all such hypergraphs with fewer edges than H. Every finite block graph contains a so-called end block, a block where all but possibly one of its vertices lie in no other block. So H must contain some edge T where all but possibly one (say x) of its points lie in no other edge. Let H' denote the hypergraph we get by deleting T and all the points $T \setminus \{x\}$. By the induction hypothesis, $H' \simeq \Phi_{k-1,k}(\Delta(P'))$ for some pure poset $P' = (A, \leq)$. Let $\{a_1 < a_2 < \ldots < a_k\}$ denote some k-flag of P' containing the $(k-1)$-chain $\{a_1 < a_2 < \ldots < a_{j-1} < a_{j+1} < \ldots < a_k\}$ corresponding to the vertex x of H'. Then we add one new point b to the poset P', where $a_{j-1} < b < a_{j+1}$ together with the necessary transitivities hold. It is easy to see that we get a pure poset P with $\Phi_{k-1,k}(\Delta(P)) \simeq H$. □

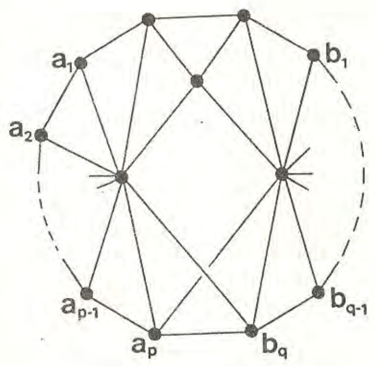

Figure 8
The graph $G_{p,q}$ — some L_3-'root' of C_{p+q+5}

Theorem 6.6 *For every $k \geq 2$ and every block graph G, the following statements are equivalent:*

(i) G is a k-flag graph,

(ii) G is a k-edge graph,

(iii) G is $K_{1,k+1}$-free.

Proof: The implication '*(i)* → *(ii)*' is trivial, and '*(ii)* → *(iii)*' follows with Corollary 3.6. Let now $G = (V, E)$ be some $K_{1,k+1}$-free block graph. Let T denote the clique hypergraph of G, having V as vertex set and all vertex sets of cliques of G as edges. Every vertex x of G lies in at most k cliques, by the assumptions for G. If the vertex x lies in p_x cliques, then we add $k - p_x$ 1-edges of the form $\{x\}$ to T to obtain a hypergraph

S, where every vertex lies in exactly k edges. Consequently, its dual S^* is k-uniform. But since S is Berge-acyclic, a self-dual notion, S^* is Berge-acyclic also. By Proposition 6.5 and Theorem 6.2, G must be a k-flag graph. □

Next we shall show that the number of $\Phi_{k-1,k}$-roots is not bounded. Look at the graph cycles C_t, where $t \geq 3$ — this are also hypergraphs. It is not too difficult to see that $\Phi_{2,3}(H_{p,q}) \simeq C_t$ for all p, q with $p + q + 5 = t$, where $H_{p,q}$ denotes the clique hypergraph of the graph $G_{p,q}$ in Figure 8. Consequently, the number of L_3-roots of a graph is also not bounded, since $L_3(G_{p,q}) \simeq C_t$ again for p, q with $p + q + 5 = t$. It is not difficult to see that $G_{p,q}$ and $G_{p',q'}$ are isomorphic only for $p = p'$ or $p = q'$.

6.2 Transformation of the problem

Let us now turn to the second problem mentioned, the simplex-edge covers that come into question. From now on we concentrate on the facet graph case only, since we need the results from Section 5. Let us assume that K is a pure $(k-1)$-dimensional simplicial complex and $G = \Psi_k(K)$, where for simplicity w. l. o. g. G has no isolated vertices. We shall describe how the cliques of G and the hyperedges of $(\Phi_{k-1,k}(K))^*$ are related, depending on their size. Any p-edge of $(\Phi_{k-1,k}(K))^*$ is induced by some $(k-1)$-edge of K and all p k-edges of K containing it. Thus any star clique of G forms a hyperedge in $(\Phi_{k-1,k}(K))^*$. Conversely and p-edge of $(\Phi_{k-1,k}(K))^*$ must also be a star clique in G, for $p \geq 3$. 2-edges of $(\Phi_{k-1,k}(K))^*$ are just the star 2-cliques and those edges of hole cliques that appear in no star clique. Recall that the partition into star and hole cliques does not depend on G alone, but also on the representation. Now $(\Phi_{k-1,k}(K))^*$ is uniquely determined up to 1-edges if this partition is known.

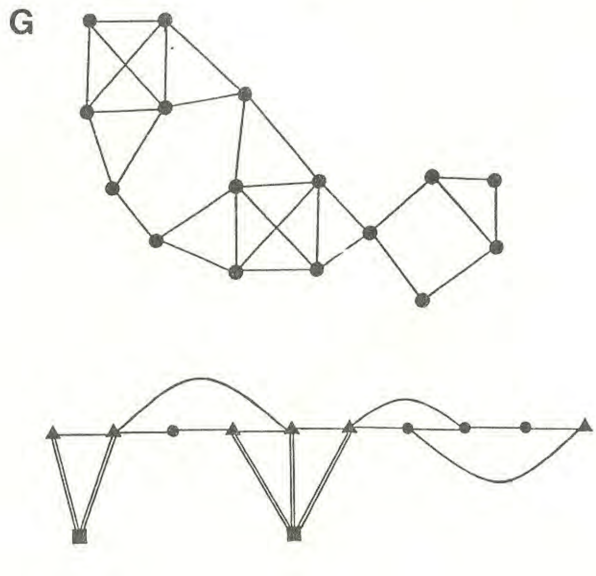

Figure 9
A graph and its weighted clique graph

But if G is connected (and we may concentrate on connected graphs, since a graph is a facet graph if and only if each of its connected components is), then $C(G)$ is also connected, and Theorem 5.1 allows us only two candidates for the partition into star cliques and hole cliques. Any 2-partition of $C(G)$ into yellow (for star cliques) and black (for hole cliques) cliques determines a star cliques – hole cliques partition candidate. Recall that all vertices corresponding to 2-cliques must be yellow. So, if there are 2-cliques, then we even have no choice at all.

We obtain the following algorithm that reduces the problem of recognizing facet graphs to that of recognizing $(k-1,k)$-hypergraphs of simplicial complexes: Two sorts of outputs are possible: Either the algorithm may say G is no facet graph, or it gives us the (at most) two possible simplex-edge covers that must be tested for being $(k-1,k)$-hypergraph. In fact we have to make the dual uniform by adding points that lie in no other edge — this corresponds to adding 1-edges in the simplex – edge cover. If these resulting k-uniform hypergraphs are no $(k-1,k)$-hypergraphs of complexes, then G is no k-facet graph. In the k-line graph case, we could even derive from the 2-partitions of $C(G)$ what values of k come into question. In general, there are only two possible values.

We use NOFACET as an abbreviation for (PRINT('G is no facet graph'); STOP).

Algorithmus 6.7 *(facet graph test)*
Input: A connected finite graph $G=(V,E)$ with $n=|V|$ vertices and $m=|E|$ edges.

1 Test whether G is (K_5-e)-free; IF not THEN NOFACET;

2 List all cliques of G and compute $C_w(G)$;

3 IF there are more than m cliques THEN NOFACET;

4 IF $g_2(G)$ is not bipartite THEN NOFACET;

5 Color all vertices of $C(G)$ corresponding to 2-cliques of G yellow;

6 Compute all possible 2-partitions $\kappa=\kappa_S \cup \kappa_H$ with all vertices of κ_S colored yellow, that obey the properties in Theorem 5.1 (DFS, for example).

7 Construct and PRINT H^* for these (at most two) 2-partitions;

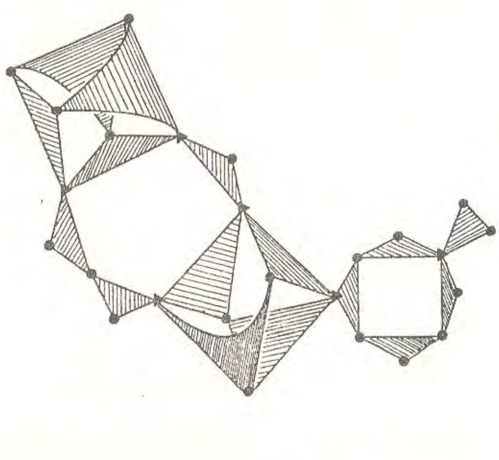

Figure 10
The hypergraph H^* for the example in Figure 9

Let us now compute the running time of the algorithms:

A straightforward implementation of step (1) requires time $O(n^5)$. Step (2) can be executed in time $O(n^4m)$ by applying the algorithm in [27] and Theorem 5.4. The running time of the other steps is obviously dominated by this.

Let us explain the method by an example. Given is the graph G of Figure 9. This graph is $(K_5 - e)$-free, so we can compute $C_w(G)$, also given in Figure 9. Vertices corresponding to 2-cliques, 3-cliques, and 4-cliques are indicated by circles, triangles, and squares, respectively. The (unique) 2-partition of $C_w(G)$ is also indicated in Figure 9.

The vertices of one of the partition classes correspond to cliques of nonhomogeneous sizes. So, if G is a k-line graph, then only for $k = 3$, where the two 4-cliques are colored black. The resulting hypergraph H^* is given in Figure 10.

It can be shown that this hypergraph is the (2,3)-hypergraph of some conformal pure 2-dimensional simplicial complex only for the triangle-hypergraph of the graph of Figure 11.

Let $E_0(J)$ denote the set of those edges of a graph J that lie in no triangle. Clearly $L_3(J) \simeq L_3(J - E_0(J))$ for any graph J.

Now $G \simeq L_k(J)$ implies $k = 3$ and $J - E_0(J) \simeq J$ except for isolated vertices. So the 'root' is a sort of unique in this case.

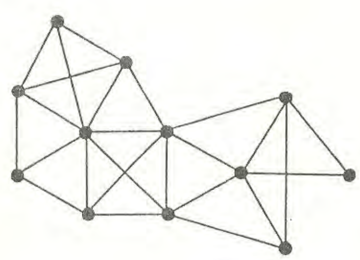

Figure 11
Some L_3-'root' for the graph in Figure 9

References

[1] H. Abels, The gallery distance of flags, to appear in Order.

[2] H. Abels, The geometry of the chamber system of a semimodular lattice, to appear in Order.

[3] B. D. Acharya, Intersection closure graphs, G. T. Newsletter 15 (March 1986).

[4] M. Aigner, A characterization problem in graph theory, J. Comb. Theory 6 (1969) 45 - 55.

[5] J.-C. Bermond, M.-C. Heydemann, D. Sotteau, Line graphs of hypergraphs I, Discrete Math. 18 (1977) 235-241.

[6] A. Björner, Homology and shellability of matroids and geometric lattices in: N. White ed., Matroid Applications, Preprint 1990.

[7] H. J. Broersma, Li Xueliang, The connectivity of the basis graph of a branching greedoid, Memorandum No.937, Universiteit Twente (1991).

[8] M. Conforti, D. G. Corneil, A. R. Mahjoub, K_i-covers I: Complexity and polytopes, Discrete Math. 58 (1986) 121-142.

[9] M. Conforti, D. G. Corneil, A. R. Mahjoub, K_i-covers II: K_i-perfect graphs, J. Graph Theory, 11 (1987) 569-587.

[10] C. R. Cook, Two characterizations of interchange graphs of complete m- partite graphs, Discrete Math. 8 (1974) 305-311.

[11] Y. Egawa, R. E. Ramos, Triangle graphs, Proceeding of the JSPS workshop Tokyo, Japan (1990) (J. Akiyama, M-J. P. Ruiz and T. Saito eds.)

[12] B. Grünbaum, Incidence patterns of graphs and complexes, in: The many Facets of Graph Theory (G. Chartrand, S. F. Kapoor eds.) LNM 110 (1970) 115-128.

[13] S. T. Hedetniemi, Graphs of (0, 1)-matrices, LNM 186 (1971) 157-171.

[14] R. L. Hemminger, L. W. Beineke, Line graphs and line digraphs, in: Selected Topics in Graph Theory I (Beineke, Wilson eds.) (1978) 271-305.

[15] M. C. Heydemann, D. Sotteau, Line-Graphs of Hypergraphs II, in: Colloq. Math. Soc. János Bolyai (1976) 567-582.

[16] C. A. Holzmann, F. Harary, On the tree graph of a matroid, SIAM J. Appl. Math. 22 (1972) 187-193.

[17] C. A. Holzmann, P. G. Norton, M. D. Tobey, A graphical representation of matroids, SIAM J. Appl. Math. 25 (1973) 618-627.

[18] B. Korte, L. Lovász, Basis graphs of greedoids and two-connectivity, Math. Programming Study 24 (1985) 158-165.

[19] J. Krausz, Démonstration nouvelle d'une théorème de whitney sur les réseaux, Mat. Fiz. Lapok 50 (1943) 75-89.

[20] V. B. Le, E. Prisner, Periodic k-line graphs, Preprint No. 306, TU Berlin (1991), Submitted.

[21] E. Marczewski, Sur deux propriétés des classes d'ensemble, Fund. Math. 33 (1945) 303 -307.

[22] S. B. Maurer, Matroid basis graphs I, J. Comb. Theory (B) 14 (1973) 216-240.

[23] S. B. Maurer, Matroid basis graphs II, J. Comb. Theory (B) 15 (1973) 121-145.

[24] E. Prisner, Infinite graphs fixed under certain graph-valued functions, (1991), Submitted.

[25] N. J. Pullman, Clique coverings of graphs IV: Algorithms, SIAM J. Comput. 13 (1984) 57-75.

[26] Y. Shibata, On the tree representation of chordal graphs, J. Graph Theory 12 (1988) 421-428.

[27] S. Tsukiama, M. Ide, M. Aiyoshi, I. Shirawaka, A new algorithm for generating all the maximal independent sets, SIAM J. Comput. 6 (1977) 505-517.

[28] Zs. Tuza, Some open problems on colorings and coverings of graphs, Graphentheorie-Tagung Oberwolfach (1990).