A Good Characterization of Cograph Contractions

Van Bang Le
FACHBEREICH INFORMATIK
UNIVERSITÄT ROSTOCK
D-18051 ROSTOCK
GERMANY
E-mail: le@informatik.uni-rostock.de

Received March 26, 1998; revised October 28, 1998

Abstract: A graph is called a *cograph contraction* if it is obtained from a cograph (a graph with no induced path on four vertices) by contracting some pairwise disjoint independent sets and then making the "contracted" vertices pairwise adjacent. Cograph contractions are perfect and generalize cographs and split graphs. This article gives a good characterization of cograph contractions, solving a problem posed by M. Hujter and Zs. Tuza. J Graph Theory 30: 309–318, 1999

Keywords: *cograph contractions; cographs; perfect graphs; 2-SAT*

1. INTRODUCTION

In [2], *complement reducible graphs*, also called *cographs*, are defined recursively as follows:

(i) A single vertex graph is a cograph.
(ii) If G_1 and G_2 are two (disjoint) cographs, then so is their union $G_1 \cup G_2$.
(iii) If G is a cograph, then so is its complement $\bar{G}$.

Cographs were investigated by a number of researchers (see [2], and the references given therein); it is well known that cographs are exactly those graphs having no chordless path on four vertices, also called P_4*-free graphs*.

 CCC 0364-9024/99/040309-10

Jung [9] proved that cographs are comparability graphs, hence perfect, where a graph G is called *perfect* if, for each induced subgraph G' of G, the chromatic number of G' equals the clique number of G'. See [1, 5] for more information on perfect graphs. The famous Strong Perfect Graph Conjecture due to C. Berge states that graphs without induced cycles of odd length at least five and their complements, called *Berge graphs*, are perfect. This conjecture is still open, and one of the main attempts to prove the conjecture consists of finding larger and larger classes of perfect Berge graphs.

In their recent study on precoloring extensions, Hujter and Tuza [8] obtained a method for generating a larger class of perfect graphs from a suitable one as follows. Let H be a graph and let $S_1, \dots, S_t$ $(t \geq 1)$ be some mutually disjoint, nonempty independent sets in H. The graph H^* is obtained from H by replacing $S_1, \dots, S_t$ by new vertices $q_1, \dots, q_t$ and joining q_i $(1 \leq i \leq t)$ to q_j $(j \neq i)$, and also joining q_i to all vertices in $V(H) - (S_1 \cup \cdots \cup S_t)$, which were adjacent to at least one vertex in S_i (see Fig. 1). Notice that, in case $t = 1$ and S_1 consisting of exactly one vertex, the contraction does not change the graph. Hujter and Tuza then proved that if H is perfect and satisfies additional conditions (in terms of precoloring extension), then H^* is a perfect graph. As they noted, their most interesting result in generating perfect graphs in this way is the case when H is a cograph. In this case, no additional condition on H is required: If H is a cograph, then H^* is perfect.

A graph G is called a *cograph contraction*, if there exists a cograph H and some pairwise disjoint independent sets in H for which $G = H^*$ holds (see Fig. 1). With this notion, the result of Hujter and Tuza states that cograph contractions are perfect. Clearly, all cographs are cograph contractions and, as the main theorem (Theorem 3.1) will imply, all split graphs are also cograph contractions. In [8] they posed the characterization problem of cograph contractions. This article solves this problem and is organized as follows. Section 2 presents two necessary conditions for cograph contractions. These conditions imply that cograph contractions are weakly triangulated graphs, hence perfect, improving Hujter and Tuza's result. Sections 3, 4, and 5 deal with the characterization and its proof. Our characterization leads to a polynomial reduction for the recognition problem of cograph contractions to the problem 2-SAT. Thus, cograph contractions can be recognized in polynomial time. Section 6 contains further discussions.

All graphs considered are finite, undirected, and have no loops or multiple edges. For a vertex v of a graph $G, N_G(v)$ denotes the set of all vertices in G adjacent to

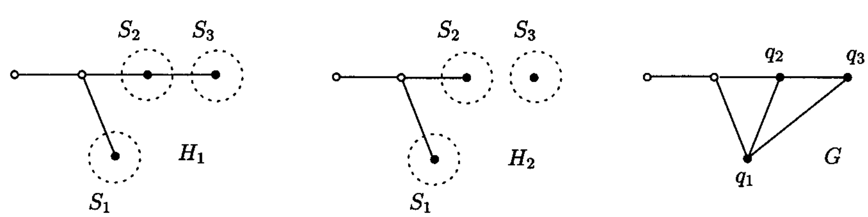

FIGURE 1. $G = (H_1)^* = (H_2)^*$. Since H_2 is a cograph, G is a cograph contraction.

v. For a set A of vertices, set $N_G(A) = \cup_{a \in A} N_G(a) - A$. When there can arise no confusion, we simply write $N(v)$ for $N_G(v)$ and $N(A)$ for $N_G(A)$. We often write $P_m = x_1 \cdots x_m$ and $C_m = x_1 \cdots x_m x_1$ for the induced path on vertices $x_1, \ldots, x_m$ with edges $x_i x_{i+1}$ $(1 \leq i < m)$, respectively, the induced cycle on vertices $x_1, \ldots, x_m$ with edges $x_1 x_m, x_i x_{i+1}$ $(1 \leq i < m)$. The *end-points* of that P_m are x_1 and x_m, and the *mid-points* are the vertices $x_i, i \neq 1, m$. The *mid-points* in $\overline{P_m}$, the complement of P_m, are the end-points of P_m. The following facts are clear and will be used in the sequel without reference:

- In a connected cograph, two nonadjacent vertices have a common neighbor.
- All induced subgraphs of a cograph contraction are also cograph contractions.

2. NECESSARY CONDITIONS

In this section, H denotes a cograph and $S_1, \ldots, S_t$ are pairwise disjoint, nonempty independent sets in H. Let $q_1, \ldots, q_t$ be the vertices in $G = H^*$ corresponding to the sets $S_1, \ldots, S_t$, respectively. By definition, $q_1, \ldots, q_t$ induce a clique Q in H^*. The first condition is as follows.

P_4-Condition. Each induced P_4 in G has at least one mid-point in Q.

Proof. Since $G - Q = H - (S_1 \cup \cdots \cup S_t)$, and the latter is a cograph, $G - Q$ has no (induced) P_4. So every P_4 of G must meet Q. If the $P_4 = abcd$ has an end-point in Q, say $a = q_i$ for some $1 \leq i \leq t$, then c and d cannot belong to the clique Q. Thus, b must belong to Q, otherwise $sbcd$ would be an induced P_4 in H for some $s \in S_i$. ∎

The next condition is less obvious.

$\overline{P_5}$-Condition. Each induced $\overline{P_5}$ in G has both mid-points in Q.

Proof. Consider the $\overline{P_5} = (\{u, v, w, x, y\}, \{uv, vw, wx, xy, yu, vx\})$ in G. Thus, v and x are the mid-points of the $\overline{P_5}$. By the P_4-Condition, at least one of x, y and at least one of u, v must be in Q. Moreover, $x \in Q$ if and only if $v \in Q$, and $u \in Q$ if and only if $y \in Q$. Hence, we are done if $x \in Q$. We will see that the case $u, y \in Q$ is impossible. For if $u = q_i, y = q_j$, then $v, w, x \in H - (S_1 \cup \cdots \cup S_t)$, and there exist some $s_i \in S_i, s_j \in S_j$ such that either $s_i v x s_j$ (if $s_i s_j \notin E(H)$) or $w v s_i s_j$ (otherwise) is an induced P_4 in H. In each case, we get a contradiction because H is a cograph. ∎

Graphs without induced C_ℓ and $\overline{C_\ell}$ $(\ell \geq 5)$ are called *weakly triangulated*. In [6] it is shown that weakly triangulated graphs are perfect.

Corollary 2.1. *Cograph contractions are weakly triangulated graphs, hence perfect.*

Proof. The cycles C_ℓ's $(\ell \geq 5)$ do not have any clique satisfying the P_4-Condition. The graphs $\overline{P_6}$ and $\overline{C_6}$ do not have any clique satisfying the $\overline{P_5}$-Condition. Hence, cograph contractions do not have any $C_\ell, \overline{C_\ell}$ $(\ell \geq 5)$ as an induced subgraph. Therefore, they are weakly triangulated. ∎

3. CHARACTERIZATION

The main result of this article is as follows.

Theorem 3.1. *A graph is a cograph contraction if and only if it has a clique satisfying the P_4-Condition and the $\overline{P_5}$-Condition.*
This characterization yields a polynomial time recognition algorithm for cograph contractions. By the theorem, recognizing cograph contractions means deciding whether a given graph contains a clique satisfying the P_4- and the $\overline{P_5}$-Conditions. The latter can be done in polynomial time using an idea in [7]. For a given graph G, we create an instance of 2-SAT as follows:

- The boolean variables are the vertices of G,
- for all nonadjacent vertices a, b of G, $(\bar{a} \vee \bar{b})$ is a clause, the *non-edge-clause* for a, b,
- for all P_4 of G having midpoints a and b, $(a \vee b)$ is a clause, the P_4*-clause* for that P_4,
- for all $\overline{P_5}$ of G having midpoints a and b, $(a \vee a)$ and $(b \vee b)$ are two clauses, the $\overline{P_5}$*-clauses* for that $\overline{P_5}$.

Our 2-SAT boolean expression is the product of all non-edge-clauses, all P_4-clauses, and all $\overline{P_5}$-clauses. Since the total number of P_4s, $\overline{P_5}$s, and pairs of nonadjacent vertices in G is bounded by $\mathcal{O}(n^5)$ $(n$ is the number of vertices of $G)$, the above reduction can be obtained in polynomial time. Now, it is easy to see that G has a clique satisfying the P_4- and the $\overline{P_5}$-Conditions if and only if our 2-SAT instance is satisfiable. Moreover, the true vertices in a satisfying assignment form a "good" clique in G. Since 2-SAT belongs to **P** (see, for example, [3, 4]), cograph contractions can be recognized in polynomial time.

Our proof of the theorem, however, is constructive. Given a graph G with a clique Q satisfying the P_4- and $\overline{P_5}$-Conditions, we will construct, efficiently, a cograph H together with $|Q|$ pairwise disjoint independent sets $S_1, \ldots, S_{|Q|}$, of H such that $G = H^*$, with respect to these independent sets S_i's.

4. CONSTRUCTION

We are given a graph G together with a clique $Q = \{q_1, \ldots, q_t\}$ that satisfies the P_4- and the $\overline{P_5}$-Conditions in G. Let R be the set $G - N(q_1) - \{q_1\}$. We partition the set of the neighbors of q_1 outside Q into two disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ as follows.

$$\mathcal{A} = \{x : x \in N(q_1) - Q \text{ adjacent to no vertex in } R\},$$

$$\mathcal{B} = \{y : y \in N(q_1) - Q \text{ adjacent to some vertex in } R\}.$$

Thus, $N(q_1) - Q = \mathcal{A} \cup \mathcal{B}$. By definition of $\mathcal{B}$, there exist a (smallest) number k and vertices $r_1, \ldots, r_k$ in R such that $\mathcal{B} \subseteq N(r_1) \cup \cdots \cup N(r_k)$. By setting

$$B_i := \mathcal{B} \cap N(r_i),$$

we get

$$\mathcal{B} = B_1 \cup \cdots \cup B_k.$$

Notice that, by the minimality of k, none of the B_i is properly contained in another. We now replace the vertex q_1 of G by the independent set (of new vertices)

$$S_1 = \{s_1, \ldots, s_k\} \cup \{s_A: A \text{ is a component of } \mathcal{A}\}, \text{ if } \mathcal{A} \text{ or } \mathcal{B} \text{ is nonempty,}$$
$$S_1 = \{q_1\} \text{ otherwise.}$$

Set $Q' = Q - \{q_1\}$. The edges between S_1 and vertices in $N(q_1)$ are defined by the following rules.

Rule 1. For each $1 \le i \le k, s_i$ is adjacent to all vertices in B_i and to all vertices in $N(r_i) \cap Q'$.

Rule 2. For each component A of $\mathcal{A}, s_A$ is adjacent to all vertices in A and to all vertices in $N(A) \cap (\mathcal{B} \cup Q')$.

The construction is illustrated in Fig. 2. Let G' denote the graph obtained; clearly, G' can be constructed in polynomial time.

In the next section we will show the following.

Reduction Lemma. *In G', the clique Q' satisfies the P_4-Condition and the $\overline{P_5}$-Condition. Moreover, $G = (G')^*$ with respect to the independent sets S_1, $\{q_2\}, \ldots, \{q_t\}$.*

From the Reduction Lemma, Theorem 3.1 follows by repeating this construction for $G := G'$ and $Q := Q'$ until all vertices q_i of Q are replaced by the independent

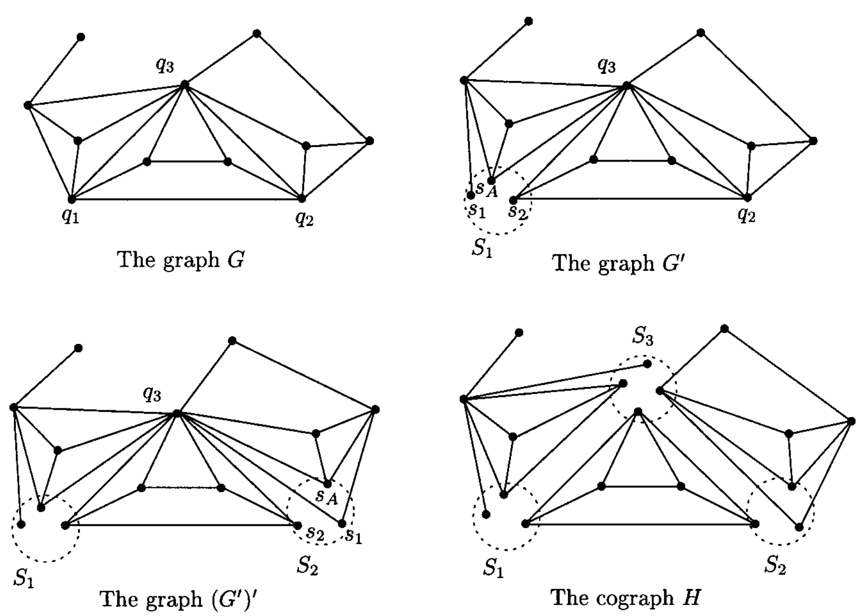

FIGURE 2. The construction.

sets S_i. Obviously, the final graph H is P_4-free and $H^* = G$ with respect to the independent sets $S_1, \ldots, S_t$.

We now give some key observations on the construction that will be helpful in proving the Reduction Lemma. The first one is quite clear.

Observation 4.1. *For all components A of $\mathcal{A}$, all vertices in A have exactly one neighbor in S_1, namely s_A. In particular, two vertices in $\mathcal{A}$ have a common neighbor in S_1 if and only if they belong to the same component of $\mathcal{A}$.* ∎

The next observations are less clear. In the proofs, we use the following term: A *bad* P_4 (a *bad* $\overline{P_5}$) in G does not have a mid-point (respectively, both mid-points) in Q. Of course, there is no bad P_4 and no bad $\overline{P_5}$ in G.

Observation 4.2. *Let x and y be nonadjacent vertices in $N(q_1)$ having a common neighbor in R. If $y \notin Q'$, then every vertex in $R \cup S_1$ adjacent to y is also adjacent to x.*

Proof. Let $y \notin Q'$ and let r be a common neighbor of x and y in R. Assume that a vertex $r' \in R$ is adjacent to y but not to x. Then $xryr'$ is a bad P_4 in G (if rr' is not an edge), or else r, r', x, y, and q_1 induce a bad $\overline{P_5}$ in G. Thus, we have shown that all vertices in R adjacent to y are also adjacent to x. Considering the r_i, this means by construction rule 1 that all s_i $(1 \le i \le k)$, adjacent to y, are also adjacent to x. To complete the proof, consider a vertex s_A in S adjacent to y. Then $y \in N(A) \cap \mathcal{B}$. If s_A is adjacent to x, we are done. Therefore, assume that s_A and x are nonadjacent. By construction rule 2, $x \notin A \cup N(A)$. Now let $a \in A$ be a neighbor of y. By definition of $\mathcal{A}$, ra is not an edge and $a \notin Q'$. As $x \notin A \cup N(A)$, xa is also not an edge. But then $xrya$ is a bad P_4 in G. This contradiction completes the proof of Observation 4.2. ∎

Observation 4.3. *Let A be a component of $\mathcal{A}$, and let $x, y \in A \cup N(A)$ be two nonadjacent vertices. Then there is a common neighbor of x and y in A.*

Proof. Since x and y are nonadjacent, at least one of them does not belong to Q', $y \notin Q'$, say. Let $a_x \in A$ be a neighbor of x. We may assume that a_x and y are nonadjacent, otherwise we are done. Now, $A \cup \{y\}$ induces a connected cograph (any P_4 in $A \cup \{y\}$ would be a bad P_4 in G, because $y \notin Q'$), there is a vertex $a \in A$ adjacent to both a_x and y. If a is nonadjacent to x, then xa_xay is a bad P_4 in G, a contradiction. Thus, a is also adjacent to x, and the observation follows. ∎

5. PROOF

We now are going to prove the Reduction Lemma; we will use the notation in the previous section. Recall that the clique Q satisfies the P_4- and the $\overline{P_5}$-Conditions in G. The case $S_1 = \{q_1\}$ (that is, $\mathcal{A} = \mathcal{B} = \emptyset$) is trivial.

An induced P_4 in G' is said to be *bad* if it has no mid-point in Q'.

Fact 5.1. *Q' satisfies the P_4-Condition in G'.*

Proof. We have to show that in G' there is no bad P_4. Suppose the contrary and let P be a bad P_4 in G'. Note that $P \cap S_1 \neq \emptyset$; otherwise, P would be a bad P_4 in G. The proof splits into two cases; in both cases, we will get a contradiction.

Case 1. P has a mid-point in S_1.

Let s be the mid-point of P in S_1. Write $P = xsyz$. Since P is bad in $G', y \in N(q_1) - Q'$. Furthermore, as Q' is a clique, at most one of x and z may be in Q'.

Subcase 1.1. $s = s_i$ for some $1 \leq i \leq k$.

In this case, x and y are adjacent to r_i. Since $y \notin Q$ and z is nonadjacent to x, Observation 4.2 implies that $z \notin R \cup S_1$. Thus, $z \in N(q_1)$, and r_i is nonadjacent to z, because $s_i z$ is not an edge (construction rule 1). Since at most one of x and z lies in Q', xr_iyz is a bad P_4 in G, a contradiction and Subcase 1.1 is settled.

Subcase 1.2. $s = s_A$ for some component A of $\mathcal{A}$.

By construction rule 2, $x, y \in A \cup N(A)$. By Observation 4.3, there exists a common neighbor $a \in A$ of x and y. As $a \in \mathcal{A}, a$ does not belong to Q'. Furthermore, z cannot be adjacent to a: This follows from definition of $\mathcal{A}$ if $z \in R$; if $z \in N(q_1)$, it follows from construction rule 2 and the fact that $s_A z$ is not an edge; if $z \in S_1$, it follows from Observation 4.1. Thus, $xayz$ is an induced P_4. Now, if $z \notin S_1$, then this P_4 is bad in G, impossible; if $z \in S_1$, then $y \in \mathcal{B}(\cap N(A))$ (by construction rule 2 and Observation 4.1). Consider a neighbor $r \in R$ of y. Since zx is not an edge, Observation 4.2 implies that r and x are nonadjacent. But then $xayr$ is a bad P_4 in G. This contradiction settles Subcase 1.2, and, hence, Case 1.

Case 2. P has an end-point in S_1.

Let s be an end-point of P in S_1. Write $P = xyzs$. Since P is bad in G', only the vertex x of P may be in Q'. By Case 1, $z \in N(q_1) - Q', y \in R \cup (N(q_1) - Q')$. Actually,

$$y \in N(q_1) - Q'.$$

For if $y \in R$, then x must belong to $N(q_1)$ (else P would be a bad P_4 in G). Then by Observation 4.2, sx would be an edge. We now discuss two cases according to whether x belongs to S_1 or not.

Subcase 2.1. $x \notin S_1$.

Suppose first that $s = s_A$ for some A. As $s_A y$ is not an edge, $z \in N(A) \cap \mathcal{B}$. Let $a \in A$ be a neighbor of z. As $s_A y$ is not an edge, a is not adjacent to y. If $x \in R$, then by definition of $\mathcal{A}, a$ is not adjacent to x. If $x \in N(q_1)$, then, since $s_A x$ is not an edge, a is also not adjacent to x. In any case, $xyza$ is a bad P_4 in G, a contradiction. Second, let $s = s_i$ for some $1 \leq i \leq k$. Then $r_i y$ is not an edge, because $s_i y$ is not (construction rule 1). r_i is also not adjacent to x: If $x \in N(q_1)$, then the edge $r_i x$ together with Observation 4.2 would imply that $s_i x$ is an edge; if $x \in R$, then the edge $r_i x$ would imply that $q_1 z r_i x$ is a bad P_4 in G. But then $r_i zyx$ is a bad induced P_4 in G. Subcase 2.1 is settled.

Subcase 2.2. $x \in S_1$.

If $s = s_A$ for some A, then $z \in N(A)$, because $s_A y$ is not an edge. Let $a \in A$ be a neighbor of z. a is not adjacent to y, because $s_A y$ is not an edge. a is also not adjacent to x by Observation 4.1. Thus, $xyza$ is a P_4 having exactly one end-point in S_1; we obtain Subcase 2.1 again. Therefore, we may assume that $s = s_i$, and by symmetry, $x = s_j$ for some $j \neq i$. Now, r_i is not adjacent to y and r_j is not adjacent to z. Hence, $r_i z y r_j$ is a bad P_4 in G (if $r_i r_j$ is not an edge), or $q_1 z r_i r_j$ is a bad P_4 in G (otherwise). In any case, we get a contradiction. This settles Subcase 2.2, hence Case 2. Thus, the proof of Fact 5.1 is completed. ∎

Fact 5.2. *Q' satisfies the $\overline{P_5}$-Condition in G'.*

Proof. In G', consider a $\overline{P_5}$ consisting of the triangle $uvwu$ and the 4-cycle $vwxyv$; thus, v and w are the mid-points of that $\overline{P_5}$. Applying Fact 5.1 for the P_4's $uvyx$ and $uwxy$, both v and w, or both x and y must belong to Q'. In the first case, we are done. Assume that x and y belong to Q'. Then $\{u, v, w\} \cap S_1 \neq \emptyset$; otherwise, the $\overline{P_5}$ considered is bad in G. Since S_1 is independent, exactly one of u, v, w belongs to S_1. Note that u, v, w are vertices outside Q'.

Case 1. $u \in S_1$.

If $u = s_i$ for some i, then r_i is not adjacent to x and y, because $s_i x$ and $s_i y$ are nonedges. Thus, r_i, v, w, x, and y induce a bad $\overline{P_5}$ in G, a contradiction. Let $u = s_A$ for some A. Then $v, w \in N(A)$ and $x, y \notin N(A)$. Let $a \in A$ be a neighbor of v. If a is not adjacent to w, then $xwva$ is a bad P_4 in G; if aw is an edge, then a, v, w, x, and y induce a bad $\overline{P_5}$ in G. These contradictions settle Case 1.

The cases $v \in S_1$ or $w \in S_1$ remain. By symmetry we only consider the first one.

Case 2. $v \in S_1$.

If $v = s_i$ for some i, then r_i is not adjacent to x, because $s_i x$ is not an edge. Therefore u, w, r_i, x, and y induce a bad $\overline{P_5}$ in G, a contradiction. Let $v = s_A$ for some A. Then $w, y \in N(A), x \notin N(A)$. By Observation 4.3, there is a vertex $a \in A$ adjacent to both w and y. Note that a is not adjacent to x, because $x \notin N(A)$. Now, as in Case 1, either $yawu$ is a bad P_4 in G (if $au \notin E(G)$), or else a, u, w, x, and y induce a bad $\overline{P_5}$ in G. This contradiction settles Case 2, and Fact 5.2 is completely proved. ∎

Moreover, it is clear by construction rules 1 and 2 that $(G')^* = G$ with respect to the independent sets $S_1, \{q_2\}, \{q_3\}, \ldots, \{q_t\}$. The Reduction Lemma follows. ∎

6. CONCLUDING REMARKS

As we have seen in Section 2, cograph contractions are weakly triangulated. So it is natural to ask which triangulated graphs are cograph contractions. By Theorem 3.1, a triangulated graph is a cograph contraction if and only if it has a clique meeting every induced P_4 in at least one mid-point. Triangulated cograph contractions can be recognized efficiently, without reduction to 2-SAT, as follows. Each triangulated graph G has at most $|V(G)|$ maximal cliques, and they can be listed in linear time (see for example [5]). The P_4-Condition will then be checked for each maximal clique.

Our construction in Section 4 yields in most cases a *disconnected* cograph H. Therefore, it would be interesting to know those cograph contractions obtained from a connected cograph. In what follows, we call a graph G a *connected-cograph contraction* if $G = H^*$ for some connected cograph H. Notice that, in this case, it is still possible that G is also obtained from a disconnected cograph as well.

The discussion below relies on the *join* operation of two graphs. Let X and Y be two disjoint graphs. The join $X + Y$ is obtained from X and Y by adding all possible edges between vertices in X and vertices in Y. As the reader may verify, $P_4 + P_4$ is an example of a connected-cograph contraction, while P_4 is not.

The following fact can be seen directly from the definition of the join- and the *-operations.

Observation 6.1. *Let H_1, H_2 be two disjoint graphs, and let $S_1^i, \ldots, S_{t_i}^i$ be some pairwise disjoint independent sets in H_i $(i = 1, 2)$. Then $S_1^1, \ldots, S_{t_1}^1, S_1^2, \ldots, S_{t_2}^2$ are pairwise disjoint independent sets of $H_1 + H_2$. Moreover, with respect to these independent sets, $(H_1 + H_2)^* = H_1^* + H_2^*$.* ∎

Our characterization of connected-cograph contractions is somewhat interesting in connection to the join-decomposition of connected cographs (see, for example, [2, 10].)

Theorem 6.1. *A graph is a connected-cograph contraction if and only if it is the join of two cograph contractions.*

Proof. First, let $G = H^*$ for some connected cograph H and some independent sets $S_1, \ldots, S_t$ in H. In [10] it is shown that $H = H_1 + H_2$ for some cographs H_1, H_2. Then, for all $S \in \{S_1, \ldots, S_t\}, S \subseteq H_1$ or else $S \subseteq H_2$, but not both. Therefore, by Observation 6.1, $G = H^* = (H_1 + H_2)^* = H_1^* + H_2^*$; with respect to those independent sets S belonging to H_i $(i = 1, 2)$.

Second, let $G = G_1 + G_2$ with two cograph contractions G_1, G_2. Consider the (disjoint) cographs H_i together with independent sets $S_1^i, \ldots, S_{t_i}^i$ $(i = 1, 2)$, for which $G_i = H_i^*$. Set $H = H_1 + H_2$; H is clearly a connected cograph. By Observation 6.1, $H^* = (H_1 + H_2)^* = H_1^* + H_2^* = G_1 + G_2 = G$. ∎

As we have seen by $P_4 + P_4$, an induced subgraph of a connected-cograph contraction need not be a connected-cograph contraction. In contrast, the class of cograph contractions is closed under taking induced subgraphs (as noted at the end of the introduction). Thus, there is a characterization of this class in terms of forbidden induced subgraphs; see the proof of Corollary 2.1 for some forbidden configurations. We are not able to find such a characterization and leave this as an open problem.

ACKNOWLEDGMENTS

The author wishes to thank one of the referees for pointing out a gap in the proofs of Facts 5.1 and 5.2 given in an earlier version of this article.

References

[1] C. Berge and V. Chvátal (Editors), Topics on perfect graphs, Annals Discrete Math 21, North–Holland, Amsterdam, 1984.

[2] D. G. Corneil, H. Lerchs, and L.S. Burlingham, Complement reducible graphs, Discrete Appl Math 10 (1981), 163–174.

[3] M. Davis and H. Putman, A computing procedure for quantification theory, J ACM 7 (1960), 201–215.

[4] S. Even, A. Itai, and A. Shamir, On the complexity of timetable and multi-commodity flow problems, SIAM J Comput 5 (1976), 691–703.

[5] M. C. Golumbic, Algorithmic graph theory and perfect graphs, Academic, New York, 1980.

[6] R. B. Hayward, Weakly triangulated graphs, J Combin Theory (B) 39 (1985), 200–209.

[7] C. T. Hoàng and V.B. Le, On P_4-transversals of perfect graphs, to appear.

[8] M. Hujter and Zs. Tuza, Precoloring extensions III: Classes of perfect graphs, Combin Prob Comput 5 (1996), 35–56.

[9] H. A. Jung, On a class of posets and the corresponding comparability graphs, J Combin Theory (B) 24 (1978), 125–133.

[10] D. Seinsche, On a property of the class of n-colorable graphs, J Combin Theory (B) 16 (1974), 191–193.