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Efficient robust algorithms for the Maximum Weight Stable Set Problem in chair-free graph classes

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Abstract

Modular decomposition of graphs is a powerful tool for designing efficient algorithms for problems on graphs such as Maximum Weight Stable Set (MWS) and Maximum Weight Clique. Using this tool we obtain $O(n \cdot m)$ time algorithms for MWS on chair- and xbull-free graphs which considerably extend an earlier result on bull- and chair-free graphs by De Simone and Sassano (the *chair* is the graph with vertices a, b, c, d, e and edges ab, bc, cd, be, and the *xbull* is the graph with vertices a, b, c, d, e, f and edges ab, bc, cd, de, bf, cf). Moreover, our algorithm is robust in the sense that we do not have to check in advance whether the input graphs are indeed chair- and xbull-free.

Keywords: Efficient graph algorithms; Robust graph algorithms; Modular decomposition; Maximum Weight Stable Set Problem on graphs; Graph algorithms

1. Introduction

A vertex set in a graph is *stable* if its elements are pairwise nonadjacent. The *Maximum* (*Weight*) *Stable Set* (MS, respectively, MWS for short) Problem asks for a maximum (vertex weight) stable set in the given graph. The M(W)S problem is a basic problem occurring in many models in Computer Science and Operations Research. It remains NP-complete even for triangle-free graphs [23], for planar graphs of degree three [14] and for $(K_{1,4}, K_4 - e)$-free graphs [7], whereas it is solvable in polynomial time for claw-free graphs [21,24]. On the other hand, its complexity is unknown for P_5-free graphs. This recently led to the investigation of a variety of graph classes defined by forbidding some small subgraphs. For example, in [19], as an application of the so-called conic reduction, Lozin gives an $O(n^4)$ time solution for the MS problem assuming that a butterfly-, chair-, co-P-, and gem-free input graph is given. Here the butterfly is the graph consisting of five vertices a, b, c, d, e with edges ab, ac, bc, cd, ce, de. See Fig. 1 for the chair, co-P and gem. In [28], Lozin's result has been improved to a polynomial time solution (which seems to be $O(n^2m)$) for the MWS problem assuming that a chair-, co-P-, and gem-free input graph is given. In [6], it is shown that the class of chair-, co-P-, and gem-free graphs has bounded clique-width, and MWS and

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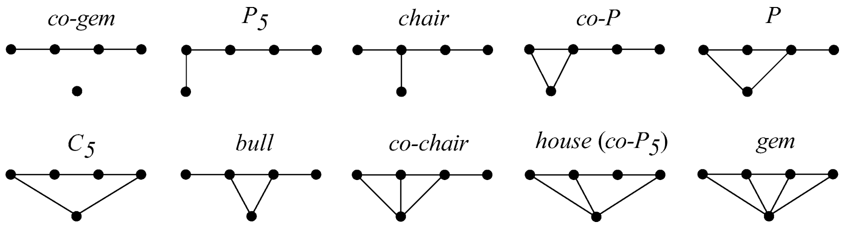

Fig. 1. All one-vertex extensions of a P_4.

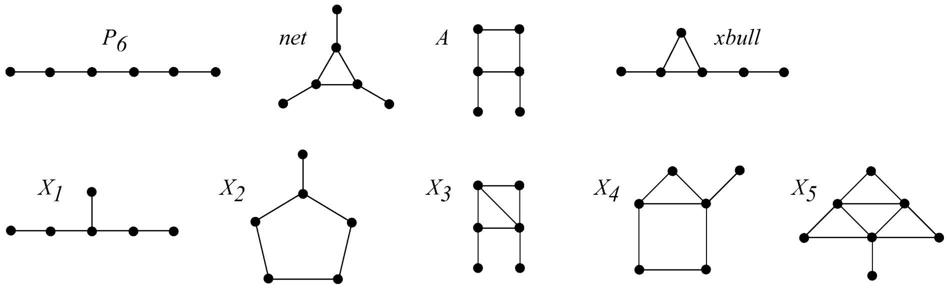

Fig. 2. Minimal prime extensions of a co-gem.

many other problems have linear-time solutions on this class.

Recently, based on Minty's algorithm for claw-free graphs [21], Alekseev [1] gave a polynomial time algorithm for the MWS problem on chair-free graphs (Minty's algorithm is included as a procedure in Alekseev's algorithm and leads to the time bound n times the complexity of the MWS problem for claw-free graphs).

We will show the following result: The structure of prime (chair, xbull)-free graphs containing a co-gem is extremely simple leading to a $O(m \cdot n)$ time algorithm for the MWS problem on this graph class since for co-gem-free graphs, the MWS problem can be solved in a straightforward way. The xbull (see Fig. 2) contains the P_5, bull and co-P as induced subgraphs; thus, our result implies such an algorithm for the classes of (chair, co-P)-free graphs, (chair, P_5)-free graphs and (chair, bull)-free graphs as well. Comparing with the above mentioned results of [1,19, 28], our algorithm is much simpler, much faster and robust. In particular, Alekseev's MWS algorithm for chair-free graphs does not work for any input graph G containing a chair. Our algorithm, however, may compute $\alpha_{\mathrm{w}}(G)$ correctly also for some input graphs containing a chair; the smallest example is the chair itself. Moreover, we extend a result by De Simone and Sassano [13] on bull- and chair-free graphs.

Following Spinrad [25], a *robust algorithm* A for a graph class $\mathcal{C}$ and an algorithmic problem Π works as follows: Given the input graph G,

- if $G \in \mathcal{C}$ then A solves the problem Π correctly,
- if $G \notin \mathcal{C}$ then A either solves Π correctly or gives a proof that $G \notin \mathcal{C}$.

Thus, a robust algorithm delivers a correct answer in any case and solves the problem for every graph in $\mathcal{C}$ without the need of recognizing $\mathcal{C}$. This is of crucial importance whenever the recognition time bound for $\mathcal{C}$ is worse than solving the algorithmic problem on the graph class; in extreme cases, the recognition problem can be NP-complete while the algorithmic problem can have a linear time solution.

2. Terminology

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $n \geqslant 3$, $|E| = m$. G is called *co-connected* if its complement $\overline{G}$ is connected.

The edges between two disjoint vertex sets X, Y form a *join* (*co-join*) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. Let $X \textcircled{1} Y$ ($X \textcircled{0} Y$) denote the corresponding join (co-join) operation between X and Y.

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M. A module is *trivial* if it is either the empty set, a one-vertex set or the entire vertex set V. A graph is *prime* if it contains only trivial modules. The notion of modules plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms—see, e.g., [22] for modular decomposition of discrete structures and its algorithmic use.

A nontrivial module M, also called *homogeneous set*, is *maximal* if no other nontrivial module properly contains M. It is well known that in a connected and co-connected graph G, the maximal nontrivial modules are pairwise disjoint which means that every vertex is contained in at most one maximal nontrivial module. The existence and uniqueness of the *modular decomposition tree* is based on this property, and recently linear time algorithms were designed to determine this tree—see [20,11,12]. The tree contains the vertices of the graph as its leaves, and the internal nodes are of three types: they represent a join or co-join operation, or a prime subgraph.

As already mentioned, a vertex set $U \subseteq V$ is *stable* (sometimes called *independent*) in G if the vertices in U are pairwise nonadjacent. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$. For a vertex weight function w on V, let $\alpha_{\mathrm{w}}(G)$ denote the maximum weight sum of a stable set in G and let $\omega_{\mathrm{w}}(G) := \alpha_{\mathrm{w}}(\overline{G})$ denote the maximum weight of a clique in G. If $\mathrm{w}(v) = 1$ for all vertices v then we omit the index w. Note that for all classes on which the MWS problem can be efficiently solved, the Maximum Weight Clique (MWC) Problem can be efficiently solved on the complement class.

It is well known that the MWS problem is solvable in polynomial time for a graph class whenever it is solvable in polynomial time for the prime graphs belonging to this class since $\alpha_{\mathrm{w}}(G_1 \textcircled{1} G_2) = \max(\alpha_{\mathrm{w}}(G_1), \alpha_{\mathrm{w}}(G_2))$, and $\alpha_{\mathrm{w}}(G_1 \textcircled{0} G_2) = \alpha_{\mathrm{w}}(G_1) + \alpha_{\mathrm{w}}(G_2)$. This immediately leads to polynomial time solutions for the MWS problem using the modular decomposition tree in a bottom-up way for computing $\alpha_{\mathrm{w}}(G)$. The same can be done for the MWC Problem.

For $k \geqslant 1$, let P_k denote an induced path with k vertices and $k-1$ edges, and for $k \geqslant 3$, let C_k denote an induced cycle with k vertices and k edges. Clearly, the MWS and the MWC problems can be solved in linear time on paths and cycles. Another class of graphs we will use later consists of the so-called spiders. A graph G is a *thin spider* if G is partitionable into a clique C and a stable set S with $|C| = |S|$ or $|C| = |S| + 1$ such that the edges between C and S form a matching and at most one vertex in C is not covered by the matching (the unmatched vertex is called the *head* of the spider). G is a *thick spider* if it is the complement of a thin spider. A *spider* is a thin spider or a thick spider. Spiders, as they are called in [17], have been introduced in [15] under the name *turtles* (see also [16]) in the context of P_4-sparse graphs. It is known (and easy to see) that recognizing spiders and solving the MWS as well as the MWC problem on spiders can be done in linear time; see also [17,18]. Let $\mathcal{F}$ denote a set of graphs. A graph G is $\mathcal{F}$-*free* if none of its induced subgraphs is isomorphic to a graph in $\mathcal{F}$.

For a vertex v of the graph $G = (V, E)$, let $N(v) := \{u \in V \mid uv \in E\}$ denote the *neighborhood* of v and let $\overline{N}(v) = V \setminus (N(v) \cup v)$. For $U \subseteq V$ let $G[U]$ denote the subgraph of G induced by U.

3. The Co-Gem Lemma

Note that for every graph $G = (V, E)$,

$$\alpha_{\mathrm{w}}(G) = \max\bigl\{\mathrm{w}(v) + \alpha_{\mathrm{w}}\bigl(G[\overline{N}(v)]\bigr):\ v \in V\bigr\}$$

holds. Thus, for co-gem-free graphs, the MWS problem can be solved in time $O(m \cdot n)$ as follows: Determine $\alpha_{\mathrm{w}}(G)$ using the fact that for every vertex v, $G[\overline{N}(v)]$ is P_4-free, and thus $\alpha_{\mathrm{w}}(G[\overline{N}(v)])$ can be computed in linear time [8–10] (see [5] for more information on P_4-free graphs).

Hence, when solving the MWS problem one can restrict the input graphs to prime graphs containing an induced co-gem. Lemma 1 describes all minimal prime extensions of the co-gem which is a useful lemma when considering the MWS problem in certain graph classes as we will see in the next section. Lemma 1 was given in [4] with a long proof using a rather complicated technique described in [27] and in [3] using a lemma from [26].

We present here a direct, short, and simple proof that also makes the paper self-contained.

Lemma 1 (The Co-Gem Lemma). *Every prime graph containing an induced co-gem contains one of the graphs in Fig.* 2 *as an induced subgraph.*

Proof. Let g denote the P_4 of an induced co-gem in the prime graph $G=(V,E)$. Partition the vertex set of $G \setminus g$ into three sets:

$$\mathcal{A}=\{x\in V\setminus g \mid x \textcircled{1} g\},$$
$$\mathcal{B}=\{x\in V\setminus g \mid x \textcircled{0} g\},$$
$$\mathcal{C}=V\setminus(g\cup\mathcal{A}\cup\mathcal{B}).$$

Note that $\mathcal{B}\neq\emptyset$ because it contains the isolated vertex of the co-gem, and $\mathcal{C}\neq\emptyset$ because otherwise g would be a homogeneous set in G.

Suppose there is at least one edge between $\mathcal{B}$ and $\mathcal{C}$, say between $b\in\mathcal{B}$ and $c\in\mathcal{C}$. All possible combinations of edges between c and g lead to one of the graphs P_6, X_1, xbull, X_2, X_3, X_4, A, *net* from Fig. 2. So we may assume that

$$\mathcal{B} \textcircled{0} \mathcal{C}$$

for otherwise we are done. Since $\mathcal{B}\neq\emptyset$ and G is connected, there is a vertex b in $\mathcal{B}$ that is adjacent to a vertex a in $\mathcal{A}$. Let $c\in\mathcal{C}$ such that $ca\notin E$. All possible combinations of edges between c and g contain one of the graphs X_3, X_4, X_5 from Fig. 2 as an induced subgraph. So partition set $\mathcal{A}$ into two subsets

$$A_0=\{x\in\mathcal{A}\mid x \textcircled{0} \mathcal{B}\} \quad \text{and}$$
$$A_1=\{x\in\mathcal{A}\mid \exists b\in\mathcal{B}\colon\ xb\in E\}$$

and we may assume

$$A_1 \textcircled{1} \mathcal{C}$$

for otherwise we are done. Note that for every $c\in\mathcal{C}$ there exist *non-adjacent* vertices $x_c, y_c\in g$ such that c is adjacent to x_c and non-adjacent to y_c. In particular, there exists a connected component X of $\overline{G}[\mathcal{A}\cup\mathcal{C}\cup g]$ containing $\mathcal{C}\cup g$. Then $X\cap A_1\neq\emptyset$ for otherwise X would be a homogeneous set in G.

Now, an X_4 in G can be detected as follows. Consider a shortest path in X (recall that X is a connected component in $\overline{G}$) connecting a vertex in $X\cap A_1$ and a vertex in $\mathcal{C}$ (recall that $\mathcal{C}\neq\emptyset$). Let $a_1a_2\ldots a_kc$ be such a path with $a_1\in A_1$ and $c\in\mathcal{C}$. Clearly, $k\geqslant 2$ and $a_i\in A_0$ for all $i=2,\ldots,k$. Let $b\in\mathcal{B}$ be a neighbor (in G) of a_1. If $k\geqslant 5$ then $a_1,\ldots,a_5,b$ induce an X_4 in G. If $k=4$ then a_1,a_2,a_3,a_4,c,b induce an X_4 in G. If $k=3$ then a_1,a_2,a_3,c,y_c,b induce an X_4 in G. Finally, if $k=2$ then a_1,a_2,c,x_c,y_c,b induce an X_4 in G. The Co-Gem Lemma is proved. □

4. Further tools

Let H be an arbitrary induced subgraph in a graph G. We call a vertex $v\in G-H$ a *k-vertex of H* if v has exactly k neighbors in H. Furthermore, if $V(H)=\{v_1,\ldots,v_h\}$ and $S\subseteq\{1,\ldots,h\}$ then M_S denotes the set of all $|S|$-vertices with the same neighbors v_i, $i\in S$, in H:

$$M_S=\bigl\{v\in G-H\colon\ N_H(v)=\{v_i\mid i\in S\}\bigr\}.$$

The set $M_\emptyset$ of 0-vertices is also denoted by M_0. We first prove the following:

Lemma 2. *If G is a prime* (*chair, xbull*)*-free graph containing an induced net then G is a thin spider.*

Proof. Let H be an induced net in G. As G is (chair, xbull)-free, one can easily check that:

(1) H has no 1-vertex and no 5-vertex.
(2) The two neighbors in H of any 2-vertex of H form a pendant edge of H.
(3) The three neighbors in H of any 3-vertex of H form the triangle of H or a $K_1\cup K_2$ where the K_2 belongs to the triangle.
(4) Three of the four neighbors in H of any 4-vertex of H form the triangle of H.

Now, consider a net N of G, say with vertices $v_1,\ldots,v_6$ and edges v_1v_2, v_2v_3, v_3v_4, v_2v_5, v_3v_5 and v_5v_6. We also write M_{123} as an abbreviation for $M_{\{1,2,3\}}$ and so on.

Claim 1. *N has no* 2*-vertex.*

Proof. By (2), it remains to show that $M_{12} \cup M_{34} \cup M_{56} = \emptyset$. By symmetry, we only show $M_{12} = \emptyset$.

Assume $M_{12} \neq \emptyset$ and let $M = M_{12} \cup \{v_1\}$. Since M is not a homogeneous set, there is a vertex $x \notin M$ adjacent to a vertex $y \in M$ and non-adjacent to a vertex $z \in M$. Without loss of generality, let $y = v_1$. Then, by (1)–(4),

$$x \in M_{135} \cup M_{1235} \cup M_{123456}.$$

Now, if $x \in M_{1235} \cup M_{135}$ then z, v_1, x, v_3, v_4 and v_5 induce an xbull—contradiction. If $x \in M_{123456}$ then x is a 5-vertex of the net H induced by $N - v_1 + z$, which contradicts (1). Claim 1 is proved.

Actually, Claim 1 shows that for any induced net H in G,

(5) H has no 2-vertex.

Continuing with the net N, we prove:

Claim 2. *$M_{135} \cup M_{1235} = \emptyset$, $M_{245} \cup M_{2345} = \emptyset$, and $M_{236} \cup M_{2356} = \emptyset$.*

Proof. By symmetry we only show the first part. Let $M = M_{135} \cup M_{1235} \cup \{v_2\}$. Assume that $M_{135} \cup M_{1235} \neq \emptyset$, and therefore $|M| \geqslant 2$. Since M is not a homogeneous set, there exists a vertex $x \notin M$ adjacent to a vertex $y \in M$ and non-adjacent to another vertex $z \in M$. Without loss of generality, let $y = v_2$ (otherwise, consider the net N' induced by $N - v_2 + y$ instead of N), and let H be the net induced by $N - v_2 + z$. By (1) and (5), x is a 3-, a 4- or a 6-vertex of N. Now, x cannot be a 3- or a 6-vertex of N because x would be a 2- or a 5-vertex of H, contradicting (5) or (1), respectively. Thus, assume that x is a 4-vertex of N. Then, by (4) and since $x \notin M$,

$$x \in M_{2356} \cup M_{2354}$$

but then x is a 3-vertex of H that contradicts (3). We have shown that $M = \emptyset$ and Claim 2 is proved.

In particular, Claim 2 implies that

(6) N has no 4-vertex, and every 3-vertex of N belongs to M_{235}.

Claim 3. *M_{235} ① M_{123456}.*

Proof. If $x \in M_{235}$ is non-adjacent to $y \in M_{123456}$ then x, v_1, v_4, v_5 and y induce a chair, a contradiction.

Claim 4. *N has no 6-vertex: $M_{123456} = \emptyset$.*

Proof. Assume $M_{123456} \neq \emptyset$. Then $M_0 \neq \emptyset$ otherwise, by Claim 3, $V(G) - M_{123456}$ would be a homogeneous set. Let

$$M_0^1 := \{v \in M_0 \mid v \text{ has a neighbor in } M_{235}\}$$

and

$$M_0^2 := M_0 - M_0^1.$$

Then $M_0^1 \neq \emptyset$, otherwise, by Claim 3, $V(G) - (M_{123456} \cup M_0^2)$ would be a homogeneous set. Moreover,

$$M_0^1 \text{ ① } M_{123456}$$

otherwise, by Claim 3, two non-adjacent vertices $x \in M_0^1$ and $y \in M_{123456}$ together with a neighbor of x in M_{235} and v_1, v_4 would induce a chair. Furthermore,

$$M_0^1 \text{ ⓪ } M_0^2$$

otherwise, if $x \in M_0^1$ is adjacent to $y \in M_0^2$, then x, y and a neighbor of x in M_{235}, v_1, v_2 and v_3 would induce an xbull. Thus, $V(G) - (M_{123456} \cup M_0^2)$ is a homogeneous set, a contradiction.

Claim 5. *M_{235} is a clique.*

Proof. Assume that x and y are two non-adjacent vertices in M_{235}. Let M be the connected component in $\overline{G}[M_{235}]$ containing x, y. As $V(M)$ is not a homogeneous set and $V(M)$ is co-connected, there exists a vertex $z \notin M$ adjacent to a vertex $x' \in M$ and non-adjacent to another vertex $y' \in M$ such that x' and y' are non-adjacent. By definition of M, $z \notin M_{235}$. By (1), (5), (6) and Claim 4, z must be a 0-vertex of N but then z, x', y', v_2, v_1 induce a chair, a contradiction.

Claim 6. *M_0 is a stable set.*

Proof. If not, let M be a nontrivial connected component of $G[M_0]$. Since M is not a homogeneous set, there exists a vertex $z \notin M$ adjacent to a vertex $x \in M$ and non-adjacent to a vertex $y \in M$ such that x and y are adjacent. By definition of M, $z \notin M_0$, hence $z \in M_{235}$ but then y, x, z, v_1, v_2, v_3 induce an xbull. Claim 6 is proved.

Claim 7. *Every vertex in M_{235} has at most one neighbor in M_0.*

Proof. If a vertex in M_{235} has two neighbors in M_0, then by Claim 6, there is a chair.

Claim 8. *Every vertex in M_0 has exactly one neighbor in M_{235}.*

Proof. If $x \in M_0$ has no neighbor (in M_{235}), then by Claim 6, $V(G) - x$ is a homogeneous set. If x has two neighbors then, by Claims 5, 6 and 7, $N(x)$ is a homogeneous set. Claim 8 follows.

Let $C := M_{235} \cup \{v_2, v_3, v_5\}$, and $S := M_0 \cup \{v_1, v_4, v_6\}$. By Claim 5, C is a clique, and by Claim 6, S is a stable set.

Claim 9. $|C| = |S|$ *or* $|C| = |S| + 1$.

Proof. By Claims 7 and 8, $|C| \geqslant |S|$. If $|C| \geqslant |S| + 2$ then $\{x \in C \mid x$ has no neighbor in $S\}$ would be a homogeneous set of G.

Now, (1), (5), (6) and Claims 4–9 show that G is a thin spider, completing the proof of Lemma 2. □

Lemma 3. *If G is a prime* (*chair, xbull*)*-free graph containing an induced P_6 then G is an induced path P_k, $k \geqslant 6$, or an induced cycle C_k, $k \geqslant 7$.*

Proof. We distinguish between two cases.

Case 1. G contains an induced cycle $C = C_k$ for some $k \geqslant 7$. In this case we will see that G is the cycle C. First, since G is (chair, xbull)-free, it is easy to see that

(7) C has no 1-vertex, no 2-vertex and no 4-vertex.

Next we prove:

Claim 1. *If a vertex $v \notin C$ is adjacent to three pairwise nonadjacent vertices in C, then v is adjacent to all vertices in C.*

Proof. Enumerate the vertices of C as $v_1, v_2, \ldots, v_k$ such that v is adjacent to v_1, v_i, v_j for some $2 < i < j - 1$. Denote by $C[1, i]$ ($C[i, j]$, $C[j, 1]$, respectively) the subpath of C between v_1 and v_i (between v_i and v_j, between v_j and v_1, respectively) such that $v_j \notin C[1, i]$ ($v_1 \notin C[i, j]$, $v_i \notin C[j, 1]$, respectively). As $k \geqslant 7$, at least one of these paths has at least four vertices, say $C[j, 1]$.

Now, v must be adjacent to all vertices in $C[j, 1]$: If not, let ℓ with $j < \ell \leqslant k$, be the smallest index such that v is nonadjacent to v_ℓ. Then v_ℓ, v_1, v, v_i, v_j (if $\ell = k$) or v_ℓ, $v_{\ell-1}$, v, v_1, v_i (if $\ell < k$) induce a chair.

Next, v must be adjacent to all vertices in $C[i, j]$: If not, let ℓ, $i < \ell < j$, be the smallest index such that v is nonadjacent to v_ℓ. Then v_ℓ, $v_{\ell-1}$, v, v_1, v_{j+1} induce a chair.

Similarly, v must be adjacent to all vertices in $C[1, i]$, and Claim 1 follows.

Claim 1 implies that

(8) C has no ℓ-vertex, $5 \leqslant \ell < k$.

Claim 2. *C has no 3-vertex.*

Proof. First, it is easy to see that the three neighbors of any 3-vertex are consecutive in C (otherwise there would exist a chair or an xbull). Now assume that C has a 3-vertex x. Enumerate the vertices of C as $v_1, v_2, \ldots, v_k$ such that x is adjacent to v_1, v_2, v_3. Since $\{x, v_2\}$ is not a homogeneous set of G, there is a vertex y adjacent to v_2 and nonadjacent to x (the case y is nonadjacent to v_2 and adjacent to x is similar by considering the cycle $C' = (C - v_2) \cup \{x\}$). Note that y cannot be adjacent to all vertices of C, otherwise x, v_1, y, v_4, v_6 would induce a chair. Thus, by (7) and (8), y must be a 3-vertex of C. As mentioned above, the three neighbors of y in C are consecutive in C.

Now, if y is adjacent to v_1, v_2, v_3 then x, v_1, v_k, v_{k-1} and y induce a chair. If y is adjacent to v_k, v_1, v_2 then y, v_1, v_k, v_{k-1}, x and v_3 induce an xbull. The case y is adjacent to v_2, v_3, v_4 is similar, and Claim 2 follows.

By (7), (8), Claim 2, and by primality of G, it follows that $G = C$.

Case 2. For all $k \geqslant 7$, G is C_k-free. In this case we will see that G is a path P_h for some $h \geqslant 6$. Let $P = v_1 v_2 \ldots v_h$ be a longest induced path in G. As G

has an induced path P_6, $h \geqslant 6$. Since P is a longest induced path and G is chair-free, we have

(9) P has no 1-vertex.

Next we prove:

Claim 3. *For every vertex v outside P, the neighbors of v in P are consecutive in P.*

Proof. If not, then there exist i, j such that $i + 1 < j$ and v is adjacent to v_i, v_j but nonadjacent to all $v_{j'}$, $i < j' < j$. As $h \geqslant 6$ and G is C_k-free for $k \geqslant 7$, $i \geqslant 2$ or $j \leqslant h - 1$. By symmetry we may assume that $i \geqslant 2$. Furthermore, $i + 2 \leqslant j \leqslant i + 4$.

Consider the case $j \in \{i + 3, i + 4\}$. Then v must be adjacent to all vertices v_ℓ, $\ell \leqslant i$ and $\ell \geqslant j$: If not, let $\ell < i$ be the smallest index such that v is nonadjacent to v_ℓ. If $\ell = i - 1$ then v_{i-1}, v_i, v_{i+1}, v and x_j induce a chair. If $\ell < i - 1$ then v_ℓ, $v_{\ell+1}$, $v_{\ell+2}$, v, v_j and v_{j-1} induce an xbull. The case $\ell \geqslant j$ is similar.

Moreover $j \neq i + 4$, for otherwise v_{i+1}, v_i, v_{i-1}, v, v_{i+4} and v_{i+3} would induce an xbull. Thus, $j = i + 3$. Now, if $i = 2$ then v_4, v_3, v_2, v_1, v and v_6 induce an xbull. If $i \geqslant 3$ then v_i, v_{i-2}, v, v_{i+3}, v_{i+2} induce a chair.

Consider the case $j = i + 2$. In this case we may further assume that $i \geqslant 3$. First assume that v is nonadjacent to v_{i-1}. Then v must be adjacent to v_{i-2}, otherwise v_{i-2}, v_{i-1}, v_i, v_{i+1} and v would induce a chair. Now, if $i > 3$ then v_{i-1}, v_{i-2}, v_{i-3}, v, v_{i+2}, v_{i+1} induce an xbull (if v is adjacent to v_{i-3}) or v_{i-1}, v_{i-2}, v_{i-3}, v, v_{i+2} induce a chair (if v is nonadjacent to v_{i-3}). If $i = 3$ then v_4, v_5, v_6, v and v_1 induce a chair (v is nonadjacent to v_6) or v_4, v_5, v_6, v, v_1, v_2 induce an xbull (otherwise).

The case when v is adjacent to v_{i-1} is similar and Claim 3 follows.

Claim 3 implies that

(10) P has no 4-vertex,

otherwise an xbull would exist, and

(11) P has no ℓ-vertex, $5 \leqslant \ell < h$,

otherwise a chair would exist.

Claim 4. *P has no 3-vertex.*

Proof. Assume to the contrary that x is a 3-vertex of P. By Claim 3, the three neighbors of x in P are v_i, v_{i+1}, v_{i+2} for some i. As $h \geqslant 6$, we may assume that $i \geqslant 3$.

Since $\{x, v_{i+1}\}$ is not a homogeneous set of G, there is a vertex y adjacent to x and nonadjacent to v_{i+1} (the case y is nonadjacent to x and adjacent to v_{i+1} is similar by considering the path $P' = (P - v_{i+1}) + x$). Now, if y is adjacent to v_{i+2} then by Claim 3, y is nonadjacent to all v_j, $j \leqslant i$, hence v_{i-2}, v_{i-1}, v_i, v_{i+1}, x and y induce an xbull. If y is nonadjacent to v_{i+2} then y must be adjacent to v_{i-2}, v_{i-1}, v_i since otherwise there would be a chair or an xbull.

Thus, by (10) and (11), y is nonadjacent to all other vertices of P. But now v_{i-2}, y, x, v_{i+1}, v_{i+2}, v_{i+3} (if $i = 3$) or v_{i-3}, v_{i-2}, v_{i-1}, y, x, v_{i+2} (if $i > 3$) induce an xbull. Claim 4 follows.

Claim 5. *P has no 2-vertex.*

Proof. Assume to the contrary that x is a 2-vertex of P. By Claim 3, and as G has no xbull, we may assume that v_1, v_2 are the two neighbors in P of x.

Since $\{v_1, x\}$ is not a homogeneous set of G, there is a vertex y adjacent to x and nonadjacent to v_1. Now, by (9), (10) and Claim 4, y cannot be adjacent to v_2, v_3 and v_4, hence v_1, v_2, v_3, v_4, x and y induce an xbull. Claim 5 follows.

By (9), (10), (11), Claims 4 and 5, and by primality of G, it follows that $G = P$. □

5. An efficient robust MWS algorithm on chair- and xbull-free graphs

By the Co-Gem Lemma as well as by Lemmas 2 and 3 we obtain the following theorems.

Theorem 1. *If G is a prime (chair, xbull)-free graph containing a co-gem then G is an induced path P_k, $k \geqslant 6$, or an induced cycle C_k, $k \geqslant 7$, or a thin spider.*

Proof. If G contains an induced co-gem then, by the Co-Gem Lemma and since G is (chair, xbull)-free,

Input: A prime graph $G=(V,E)$ with a vertex weight w
Output: $\alpha_{\mathrm{w}}(G)$ or "G is not (chair, xbull)-free"

1. **if** G is co-gem-free
2. **then** compute $\alpha_{\mathrm{w}}(G)=\max\{\mathrm{w}(v)+\alpha_{\mathrm{w}}(G[\overline{N}(v)]):\ v\in V\}$ and STOP
3. **else if** G is not a path and not a cycle and not a thin spider
4. **then** output "G is not (chair, xbull)-free" and STOP
5. **else** compute $\alpha_{\mathrm{w}}(G)$ in the obvious way

Algorithm `RobustMWS`.

G contains an induced P_6 or an induced net. Then Theorem 1 follows from Lemmas 2 and 3. □

Theorem 2. *If G is a prime* (*chair,* co-P)-*free graph containing a co-gem then G is an induced path P_k, $k\geqslant 6$, or an induced cycle C_k, $k\geqslant 7$, or a thin spider.*

Proof. The proof of Theorem 2 is literally the same as the one of Theorem 1. □

Theorem 3. *If G is a prime* (*chair, P_5*)-*free graph containing a co-gem then G is a thin spider.*

Proof. If G contains an induced co-gem then, by the Co-Gem Lemma and since G is (chair, P_5)-free, G contains an induced net. Then Theorem 3 follows from Lemma 2. □

Theorem 4. *If G is a prime* (*chair, bull*)-*free graph containing a co-gem then G is an induced path P_k, $k\geqslant 6$, or an induced cycle C_k, $k\geqslant 7$.*

Proof. If G contains an induced co-gem then, by the Co-Gem Lemma and since G is (chair, bull)-free, G contains an induced P_6. Then Theorem 4 follows from Lemma 3. □

Combining Theorem 1 and the $O(m\cdot n)$ time MWS algorithm for co-gem-free graphs pointed out in Section 3 we obtain the robust $O(m\cdot n)$ time Algorithm `RobustMWS` (see above) for (chair, xbull)-free graphs. As already mentioned, we may restrict our attention to *prime* input graphs.

Note that similar robust algorithms can be obtained for MWS on (chair, co-P)-free graphs by Theorem 2, on (chair, P_5)-free graphs by Theorem 3, and on (chair, bull)-free graphs by Theorem 4. Note also that already in [2] an $O(n\cdot m)$ time robust algorithm for MWS on (chair, bull)-free graphs is given improving the result of [13].

6. Concluding remarks

The $O(m\cdot n)$ time MWS algorithm for co-gem-free graphs pointed out in Section 3 suggests considering larger and larger classes for which the MWS problem can be solved in polynomial time as follows.

First, let $\mathcal{C}_0$ be the class of all P_4-free graphs. Then, for every $k\geqslant 1$, let

$$\mathcal{C}_k := \bigl\{G \mid \forall v\in G:\ G[\overline{N}(v)]\in\mathcal{C}_{k-1}\bigr\}.$$

Recall that $\overline{N}(v):=V(G)-(N(v)\cup v)$. Thus, $\mathcal{C}_1$ is exactly the class of co-gem-free graphs. Clearly, the MWS can be solved in time $O(m\cdot n^k)$ for graphs in $\mathcal{C}_k$. So, the question arises: Given a graph G, can one efficiently compute the smallest number k such that $G\in\mathcal{C}_k$? Unfortunately, this problem is co-NP-hard:

Observation 1. Given graph G and an integer k. Deciding whether G does not belong to $\mathcal{C}_k$ is NP-complete, even for triangle-free graphs and for planar graphs of maximum degree 3.

Proof. We reduce the MS problem to this problem. Given G and k, let H be the disjoint union of G and an induced P_4. We claim that

$$\alpha(G)\geqslant k \text{ if and only if } H\notin\mathcal{C}_k.$$

To see this, note first that an arbitrary graph X belongs to $\mathcal{C}_k$ if and only if X is $(kK_1\cup P_4)$-free where kK_1 denotes the union of k isolated vertices. Now, assume that $\alpha(G)\geqslant k$ and let S be a stable set of at least k vertices in G. Consider a vertex $v\in S$. Then $H[\overline{N}(v)]$ contains an induced $(k-1)K_1\cup P_4$ and therefore, $H[\overline{N}(v)]\notin\mathcal{C}_{k-1}$. Hence $H\notin\mathcal{C}_k$.

Conversely, assume that $H\notin\mathcal{C}_k$. By definition, there is a vertex $v\in H$ such that $H[\overline{N}(v)]\notin\mathcal{C}_{k-1}$. Thus, $\alpha(H[\overline{N}(v)])\geqslant(k-1)+2=k+1$ and then, $H[\overline{N}(v)]$ contains an induced $(k-1)K_1\cup P_4$, implying $\alpha(G)+2=\alpha(H)\geqslant\alpha(H[\overline{N}(v)])+1\geqslant(k+1)+1$. Hence $\alpha(G)\geqslant k$.

As already mentioned, it is known that the MS problem is NP-complete even for triangle-free graphs [23] and for planar graphs of degree 3 [14]. Now the second part of the observation follows from the definition of H. □

Another question of interest is to find (robust) algorithms for the Maximum Weight Clique problem on the classes discussed in Section 5.

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