SIAM J. DISCRETE MATH.
Vol. 17, No. 3, pp. 341–360

SPLIT-PERFECT GRAPHS: CHARACTERIZATIONS AND ALGORITHMIC USE*

ANDREAS BRANDSTÄDT† AND VAN BANG LE†

Abstract. Two graphs G and H with the same vertex set V are P_4-*isomorphic* if every four vertices $\{a, b, c, d\} \subseteq V$ induce a chordless path (denoted by P_4) in G if and only if they induce a P_4 in H. We call a graph *split-perfect* if it is P_4-isomorphic to a split graph (i.e., a graph being partitionable into a clique and a stable set). This paper characterizes the new class of split-perfect graphs using the concepts of homogeneous sets and p-connected graphs and leads to a linear time recognition algorithm for split-perfect graphs, as well as efficient algorithms for classical optimization problems on split-perfect graphs based on the primeval decomposition of graphs. The optimization results considerably extend previous ones on smaller classes such as P_4-sparse graphs, P_4-lite graphs, P_4-laden graphs, and (7,3)-graphs. Moreover, split-perfect graphs form a new subclass of brittle graphs containing the superbrittle graphs for which a new characterization is obtained leading to linear time recognition.

Key words. perfect graphs, P_4-structure of perfect graphs, graphs with P_4-structure of split graphs, perfectly orderable graphs, brittle graphs, superbrittle graphs, P_4-sparse graphs, P_4-lite graphs, P_4-laden graphs, good characterization, linear time recognition, primeval decomposition tree

AMS subject classifications. 05C17, 05C75, 05C85

DOI. 10.1137/S0895480100367676

1. Introduction. Graph decomposition is a powerful tool in designing efficient algorithms for basic algorithmic graph problems such as maximum independent set, minimum coloring, and many others. Recently, the modular, the primeval, and the homogeneous decomposition of graphs attracted much attention. The last two types of decomposition were introduced by Jamison and Olariu [42] (see also [5]) and are based on their structure theorem and the concept of P_4-connectedness. A P_4 is an induced path on four vertices. A graph $G = (V, E)$ is P_4-*connected* (p-*connected* for short) if, for every partition V_1, V_2 of V with nonempty V_1, V_2, there is a P_4 of G with vertices in V_1 and in V_2, called *crossing* P_4. It is easy to see that every graph has a unique partition into maximal induced p-connected subgraphs, called p-*connected components* (p-*components* for short), and vertices belonging to no P_4.

We follow this line of research by introducing and characterizing a new class of graphs—the *split-perfect graphs*—for which the p-connected components have a simple structure generalizing split graphs. As usual, a graph is called a *split graph* if its vertex set can be partitioned into a clique and a stable set.

The p-connected components represent the nontrivial leaves in the primeval decomposition tree, and thus some basic algorithmic problems can be solved in linear time along the primeval decomposition tree.

The primeval tree is a generalization of the *cotree* representing the structure of the well-known *cographs*, i.e., the graphs containing no induced P_4. A cograph or its complement is disconnected, and the cotree expresses this in terms of corresponding

*Received by the editors February 10, 2000; accepted for publication (in revised form) June 25, 2003; published electronically January 6, 2004. An extended abstract of this paper appeared in *Graph-Theoretic Concepts in Computer Science*, Lecture Notes in Comput. Sci. 1928, Springer–Verlag, Berlin, 2000, pp. 71–82.
http://www.siam.org/journals/sidma/17-3/36767.html
†Institut für Theoretische Informatik, Fachbereich Informatik, Universität Rostock, 18051 Rostock, Germany (ab@informatik.uni-rostock.de, le@informatik.uni-rostock.de).

cojoin and join operations. The cotree representation of a cograph is essential in solving various NP-hard problems efficiently for these graphs; see [19, 20] for more information on P_4-free graphs.

The study of P_4-free graphs has motivated considering graphs with few P_4's, such as *P_4-reducible graphs* [37, 40] (no vertex belongs to more than one P_4), *P_4-sparse graphs* [32, 33, 39, 41, 44] (no set of five vertices induces more than one P_4), *P_4-lite graphs* [38] (every set of at most six vertices induces at most two P_4's or a "spider"), and *P_4-laden graphs* [28] (every set of at most six vertices induces at most two P_4's or a split graph). Note that in this order, every graph class mentioned in this paragraph is a subclass of the next one.

Recently, Babel and Olariu [4] considered graphs in which no set of at most q vertices induces more than t P_4's, called *(q, t)-graphs*. The most interesting case is $t = q - 4$: (4,0)-graphs are exactly the P_4-free graphs, (5,1)-graphs are exactly the P_4-sparse graphs, and it turns out that P_4-lite graphs form a subclass of (7,3)-graphs. For all these graphs, nice structural results have been obtained that yield efficient solutions for classical NP-hard problems. Our new class of split-perfect graphs extends all of them.

Another motivation for studying graph classes with special P_4-structure stems from the *greedy coloring heuristic*: Define a linear order $<$ on the vertex set, and then always color the vertices along this order with the smallest available color. Chvátal [17] called $<$ a *perfect order* of G if, for each induced subgraph H of G, the greedy heuristic colors H optimally. Graphs having a perfect order are called *perfectly orderable* (see [34] for a comprehensive survey); they are NP-hard to recognize [46]. Because of the importance of perfectly orderable graphs, however, it is natural to study subclasses of such graphs which can be recognized efficiently. Such a class was suggested by Chvátal in [16]; he called a graph G *brittle* if each induced subgraph H of G contains a vertex that is not an endpoint of any P_4 in H or not a midpoint of any P_4 in H. Brittle graphs are discussed in [35, 50, 51]. Babel and Olariu [4] proved that (7,3)-graphs are brittle, and Giakoumakis [28] proved that P_4-laden graphs are brittle. A natural subclass of brittle graphs, called *superbrittle*, consists of those graphs G in which *every* vertex is not an endpoint of any P_4 in G or not a midpoint of any P_4 in G. Split graphs are superbrittle since in a split graph with clique C and stable set S, every midpoint of a P_4 is in C and every endpoint of a P_4 is in S. Superbrittle graphs are characterized in terms of forbidden induced subgraphs in [47]. We will show that our new class of split-perfect graphs is a subclass of brittle graphs, containing all superbrittle graphs. Moreover, we construct a perfect order of a split-perfect graph efficiently, and we obtain a new characterization of superbrittle graphs leading to a linear time recognition.

Yet another motivation for studying split-perfect graphs stems from the theory of perfect graphs. A graph G is called *perfect* if, for each induced subgraph H of G, the chromatic number of H equals the maximum number of pairwise adjacent vertices in H. For example, all the above-mentioned graphs are perfect. For more information on perfect graphs, see [7, 12, 29]. Recognizing perfect graphs in polynomial time is a major open problem in algorithmic graph theory.[1] Two graphs G and H with the same vertex set V are *P_4-isomorphic* if, for all subsets $S \subseteq V$, S induces a P_4 in G if and only if S induces a P_4 in H. Chvátal [18] conjectured and Reed [48] proved that two P_4-isomorphic graphs are both perfect or both imperfect. Thus, to recognize

[1] Very recently, Chudnovsky et al. [14], Chudnovsky and Seymour [15], and Cornuéjols, Liu, and Vušković [22] have announced that perfect graphs can be recognized in polynomial time.

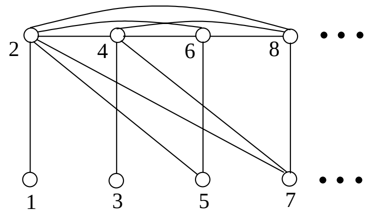

FIG. 1.1. *Elementary graphs illustrated.*

perfect graphs it is enough to recognize the P_4-structure of perfect graphs: given a 4-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$. Is there a perfect graph $G = (V, E)$ such that $S \in \mathcal{E}$ if and only if S induces a P_4 in G? This was done for the case when the perfect graph G is a tree [25, 10, 11], a block graph [8], the line graph of a bipartite graph [52], a claw-free graph [3], or a bipartite graph [2]. Note that the P_4-structure of a (not necessarily perfect) graph can be recognized in polynomial time [31].

Another question arising from Reed's theorem is the following: Which (perfect) graphs are P_4-isomorphic to a member of a given class of perfect graphs? Let $\mathcal{C}$ be a class of perfect graphs. Graphs P_4-isomorphic to a member in $\mathcal{C}$ are called $\mathcal{C}$*-perfect graphs*. By Reed's theorem, $\mathcal{C}$-perfect graphs are perfect. Moreover, they form a class of graphs which is closed under complementation and contains $\mathcal{C}$ as a subclass. Thus, it is interesting to ask the following question: Assuming that there is a polynomial time algorithm for testing membership in $\mathcal{C}$, can $\mathcal{C}$-perfect graphs be recognized in polynomial time, too? First results in this direction are good characterizations of tree-perfect graphs, forest-perfect graphs [9], and bipartite-perfect graphs [43]. This paper will give a good characterization of split-perfect graphs.

DEFINITION 1.1. *A graph is called* split-perfect *if it is P_4-isomorphic to a split graph.*

Trivial examples of split-perfect graphs are split graphs and P_4-free graphs. Nontrivial examples are induced paths $P_n = v_1 v_2 \cdots v_n$ for any integer n. To see this we need some definitions, following [9]. Let $(v_1, \ldots, v_n)$ be a vertex order of a graph G. Then $N_{>i}(v_i)$ denotes the set of all neighbors v_k of v_i with $k > i$. A vertex order $(v_1, \ldots, v_n)$ of G is said to be *elementary* if for all i

$$N_{>i}(v_i) = \begin{cases} \{v_{i+2}, v_{i+3}, \ldots, v_n\} & \text{for even } i, \\ \{v_{i+1}\} & \text{for odd } i. \end{cases}$$

Graphs having elementary orders are split graphs in which the "odd vertices" v_{2k+1} form a stable set and the "even vertices" v_{2k} form a clique. A graph is said to be *elementary* if it has an elementary order (see Figure 1.1). If the elementary graph has at least 4 vertices, then its partition into a clique and a stable set is unique and can be determined using its degree sequence. Thus, as split graphs in general [30], elementary graphs can be recognized in linear time.

Obviously, $P_n = v_1 v_2 \cdots v_n$ is P_4-isomorphic to the elementary graph consisting of the elementary order $(v_1, \ldots, v_n)$. It can be seen that, for $n \geq 7$, this elementary graph is the only split graph (up to "complementation" and "bipartite complementation") that is P_4-isomorphic to P_n. In section 4, we will extend this example to the so-called *double-split graphs*. Double-split graphs play a key role for characterizing split-perfect graphs.

In section 2, we will show that the class of split-perfect graphs contains all P_4-laden graphs and all (7,3)-graphs (hence all P_4-reducible, P_4-sparse, and P_4-lite

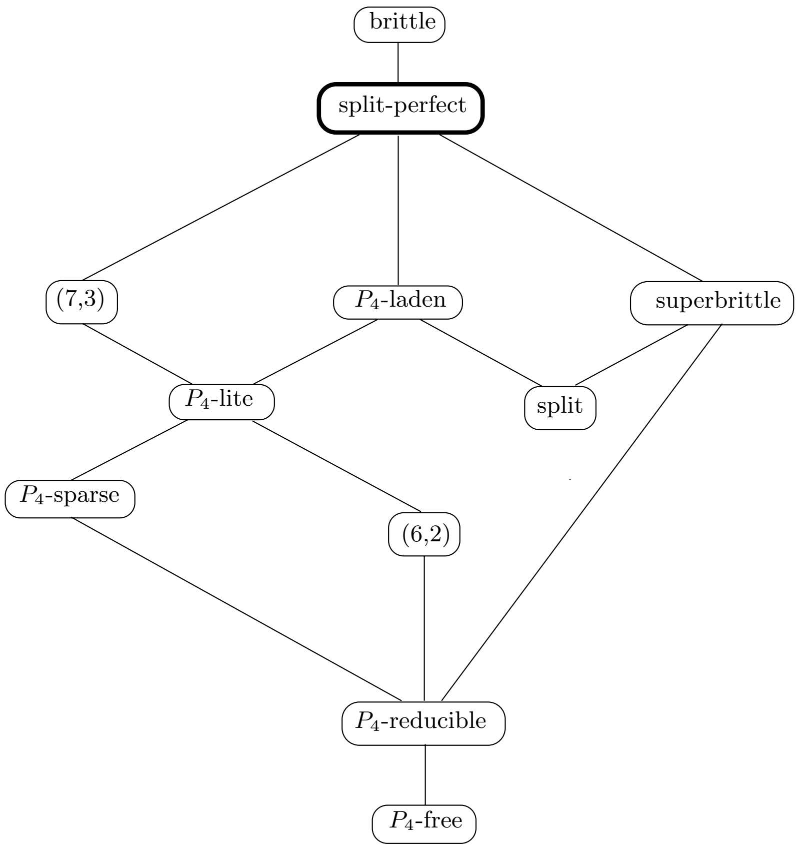

FIG. 1.2. *Relationship between graph classes.*

graphs). The relationship between the above-mentioned graph classes is shown in Figure 1.2.

In section 3, we describe forbidden induced subgraphs of split-perfect graphs, which are needed for characterizing split-perfect graphs.

In section 4, we introduce double-split graphs and show that they are split-perfect. As already mentioned, double-split graphs are of crucial importance for a good characterization of split-perfect graphs.

In section 5, we characterize split-perfect graphs in terms of forbidden subgraphs and in terms of their p-connected components: It turns out that for split-perfect graphs having no homogeneous sets, the p-connected components are double-split graphs or their complements.

In the last section, section 6, we will point out how classical optimization problems such as weighted clique number, weighted chromatic number, weighted independence number, and weighted clique cover number can be solved efficiently, in a divide and conquer manner, on split-perfect graphs using the primeval decomposition tree. These results are based on our good characterization of p-connected split-perfect graphs.

2. Preliminaries. Our notation is quite standard. The neighborhood of the vertex v in a graph G is denoted by $N_G(v)$; if the context is clear, we simply write $N(v)$. The path (respectively, cycle) on m vertices $v_1, v_2, \ldots, v_m$ with edges v_iv_{i+1}

(respectively, v_iv_{i+1} and v_1v_m) ($1 \leq i < m$) is denoted by $P_m = v_1v_2 \cdots v_m$ (respectively, $C_m = v_1v_2 \cdots v_mv_1$). The vertices v_1 and v_m are the *endpoints* of the path P_m, and for a P_4 $v_1v_2v_3v_4$, v_2 and v_3 are the *midpoints* of the P_4. Graphs containing no induced subgraphs isomorphic to a graph of a given set H of graphs are called *H-free graphs*. It is well-known that split graphs are exactly the $(C_4, \overline{C_4}, C_5)$-free graphs [26].

For convenience, we often identify sets of vertices of a graph G and the subgraphs induced by these sets in G. Thus, for $S \subseteq V(G)$, S also denotes the subgraph $G[S]$ induced by S.

A set S of at least two vertices of a graph G is called *homogeneous* if $S \neq V(G)$ and every vertex outside S is adjacent to all vertices in S or to no vertex in S. A graph is *prime* if it has at least three vertices and contains no homogeneous set. Obviously, prime graphs and their complements are connected.

A homogeneous set M is *maximal* if no other homogeneous set properly contains M. It is well known that in a connected graph G with connected complement $\overline{G}$, the maximal homogeneous sets are pairwise disjoint (see, e.g., [45]). In this case, the graph G^* obtained from G by contracting every maximal homogeneous set to a single vertex is called the *characteristic graph* of G. Clearly, G^* is prime. We shall use the following useful fact for later discussions (see Figure 3.1 for the graphs G_i).

LEMMA 2.1 (see [36]). *Every prime graph containing an induced C_4 contains an induced $\overline{P_5}$ or G_3 or G_4.*

Throughout this paper, we use the fact that, in a graph $G = (V, E)$, every homogeneous set S contains exactly one vertex of every P_4 crossing S and $V \setminus S$.

For the subsequent structure theorem of Jamison and Olariu we need the following notion: A p-component H of G is called *separable* if it has a partition into nonempty sets H_1, H_2 such that every P_4 with vertices from both H_i's has its midpoints in H_1 and its endpoints in H_2. Note that a p-connected graph is separable if and only if its characteristic graph is a split graph [42].

THEOREM 2.2 (structure theorem [42]). *For an arbitrary graph G, precisely one of the following conditions is satisfied:*

(i) *G is disconnected,*
(ii) *$\overline{G}$ is disconnected,*
(iii) *G is p-connected,*
(iv) *there is a unique proper separable p-component H of G with a partition (H_1, H_2) such that every vertex outside H is adjacent to all vertices in H_1 and nonadjacent to all vertices in H_2.*

Based on this theorem, Jamison and Olariu define the primeval decomposition, which can be described by the primeval decomposition tree and leads to efficient algorithms for a variety of problems if the p-connected components are sufficiently simple. We will show that this is the case for split-perfect graphs.

Note that dividing a graph into p-connected components can be done in linear time (see [6]). This fact together with Proposition 2.3 below allows us to restrict our attention to p-connected split-perfect graphs only.

PROPOSITION 2.3. *A graph is split-perfect if and only if each of its p-connected components is split-perfect.*

Proof. The only if part is clear. To prove the if part, let G be a graph such that each p-connected component A_i ($1 \leq i \leq m$) of G is P_4-isomorphic to a split graph B_i. Let W be the set of all vertices of G not belonging to any P_4. We now construct, inductively, a split graph H_m P_4-isomorphic to G as follows.

First, set $H_1 := B_1 \cup W$. If the split graph H_i ($1 \leq i < m$) is already constructed, then H_{i+1} is obtained from H_i and B_{i+1} by joining every vertex in the clique part of H_i and every vertex of B_{i+1} by an edge.

Clearly, H_m is a split graph. Moreover, B_i $(1 \le i \le m)$ are exactly the p-connected components of H_m. Thus, H_m is P_4-isomorphic to G. □

OBSERVATION 2.4. *Let G be split-perfect and let $H = (C_H, S_H, E_H)$ be a split graph P_4-isomorphic to G. Assume that each of the sets $\{a, b, c, u\}$ and $\{a, b, c, v\}$ induces a P_4 in G. Then exactly one of the following conditions holds:*

(i) *a, b, c induce a path P_3 in H, and u and v are both adjacent in H to an endpoint of the path $H[a, b, c]$. In particular, u and v both belong to the stable-part S_H of H.*

(ii) *The statement* (i) *holds in $\overline{H}$ instead of H. In particular, u and v both belong to the clique-part C_H of H.*

Proof. Since a, b, c, and u induce a P_4 in H, $H[a, b, c]$ must be a P_3, or else a $\overline{P_3}$. The rest follows from the fact that H is a split graph. □

PROPOSITION 2.5. *Let G be a p-connected split-perfect graph. Then every homogeneous set of G induces a P_4-free graph.*

Proof. Assume to the contrary, that there is a homogeneous set S in G which contains an induced P_4 $x_1x_2x_3x_4$. As G is p-connected, there is a crossing P_4 P to the partition S and $V(G) - S$. As S is homogeneous, P has exactly one vertex in S. Let a, b, c be the three vertices of P outside S. Since S is homogeneous, each of the sets $\{a, b, c, x_i\}$, $1 \le i \le 4$, induces a P_4 in G. Now, by Observation 2.4, if H is an arbitrary split graph P_4-isomorphic to G, then in H, x_1, x_2, x_3, x_4 are pairwise nonadjacent, or else pairwise adjacent. In particular, $H[x_1, x_2, x_3, x_4]$ cannot be a P_4, a contradiction. □

PROPOSITION 2.6. *Let G be a p-connected graph. G is split-perfect if and only if*

(i) *every homogeneous set of G induces a P_4-free graph, and*

(ii) *G^* is split-perfect.*

Proof. The necessity is clear, because of Proposition 2.5 and the fact that G^* is (isomorphic to) an induced subgraph of G. We now prove the sufficiency. Let G^* be P_4-isomorphic to a split graph H. For each vertex v of G^* let M_v be the corresponding maximal homogeneous set in G. Let H' be the graph obtained from H by replacing each vertex v by the complete graph on vertex set M_v (if v belongs to the clique part of H), respectively, be the stable set M_v (otherwise). Clearly, H' is a split graph. Since the sets M_v contain no P_4, G and H' are P_4-isomorphic (extend a P_4-isomorphism between G^* and H to one between G and H' in a natural way). □

Propositions 2.3 and 2.6 allow us to consider only p-connected split-perfect graphs without homogeneous sets.

Recall that P_4-laden graphs are those graphs in which every set of at most six vertices induces at most two P_4's or a split graph.

COROLLARY 2.7.

(i) *P_4-laden graphs are split-perfect.*

(ii) (7,3)*-graphs are split-perfect.*

Proof. To prove (i), let G be a p-connected P_4-laden graph. Then

every homogeneous set of G consisting of more than two vertices is a stable set,

otherwise, let M be a homogeneous set with at least three vertices a, b, c, where a and b are adjacent. By the p-connectedness, there is a crossing P_4 P for M and $V(G) - M$. As M is homogeneous, $|V(P) \cap M| = 1$. Now, $(V(P) - M) \cup \{a, b, c\}$ consists of exactly six vertices, induces three P_4's, but does not induce a split graph, a contradiction.

Now, it was proved in [28, Theorem 10] that

$$G^* \text{ is a } P_5 \text{ or } \overline{P_5} \text{ or a split graph.}$$

In particular, G^* is split-perfect and (i) follows from Propositions 2.6 and 2.3.

To prove (ii), we first show the following claims; the first one is easy to see; the second one follows from the known inclusions (6,2) $\subset$ P_4-lite $\subset$ P_4-laden and (i).

CLAIM 1. *Every graph with at most five vertices, different from the C_5, is split-perfect.* □

CLAIM 2. (6,2)*-graphs are split-perfect.* □

Now, consider a p-connected (7,3)-graph G. We have to show that G is split-perfect. It was shown in [4, Theorem 4.5] that G has at most six vertices. By Claims 1 and 2, we may assume that G has exactly six vertices and exactly three P_4's.

If G has a homogeneous set, G^* is split-perfect by Claim 1 and every homogeneous set has at most three vertices (otherwise, the p-connectedness would imply that G has four P_4's). Hence, by Proposition 2.6, G is split-perfect. So, let G have no homogeneous set.

If G or $\overline{G}$ has a P_5, say G, then (by considering the neighbors of the vertex outside the P_5) G is a P_6 or the graph with vertices v_i ($1 \le i \le 6$) and edges v_iv_{i+1} ($1 \le i \le 5$), v_2v_6, and v_3v_6 (otherwise G has a homogeneous set or four P_4's). In each case, G is split-perfect.

If G is $(P_5, \overline{P_5})$-free, then G cannot contain an induced C_4 or $\overline{C_4}$. Otherwise, by Lemma 2.1, G would contain a G_3, $\overline{G_3}$, G_4, or $\overline{G_4}$, but each of these graphs has more than three P_4's, a contradiction. Thus, G is $(C_4, \overline{C_4}, C_5)$-free; i.e., G is a split graph and (ii) follows. □

3. Forbidden induced subgraphs for split-perfect graphs. As a consequence of Observation 2.4, we give a list of forbidden induced subgraphs of split-perfect graphs: These are the induced cycles C_k of length $k \ge 5$, the graphs G_i ($1 \le i \le 8$) shown in Figure 3.1, and their complements. It turns out (Theorem 5.1) that these forbidden induced graphs characterize prime split-perfect graphs.

We need some notions. Let G and G' be two graphs with the same vertex set. An induced P_4 in G is *bad* if its vertices do not induce a P_4 in G' (thus, P_4-isomorphic graphs do not have bad P_4's).

Another useful notion is suggested by Observation 2.4: Let G be a split-perfect graph and H a corresponding split graph having the same P_4-structure. We call the clique and the stable set of H the two *classes* of H. Two vertices x, y in G are called *equivalent* ($x \sim y$) if they are in the same class of H. Clearly, $\sim$ is an equivalence relation on the vertex set of a split-perfect graph.

Now, Observation 2.4 means that in a split-perfect graph G, vertices x and y are in the same class (i.e., $x \sim y$) if there are vertices $a, b, c \in V(G) - \{u, v\}$ such that $\{a, b, c, x\}$ and $\{a, b, c, y\}$ both induce a P_4.

Therefore, in a split-perfect graph, pairwise equivalent vertices induce a P_4-free subgraph.

Recall that a P_4 in a split graph H has its two midpoints in one class and its two endpoints in the other class. Thus, if G is P_4-isomorphic to H, then every P_4 P of G must be *balanced* with respect to H; i.e., P has exactly two vertices in one class and the other two vertices in the other class.

LEMMA 3.1. *None of the graphs $C_k, \overline{C_k}$ ($k \ge 5$), and $G_i, \overline{G_i}$ ($1 \le i \le 8$) in Figure* 3.1 *is split-perfect.*

Proof. Throughout this proof, we will extensively use the facts discussed above.

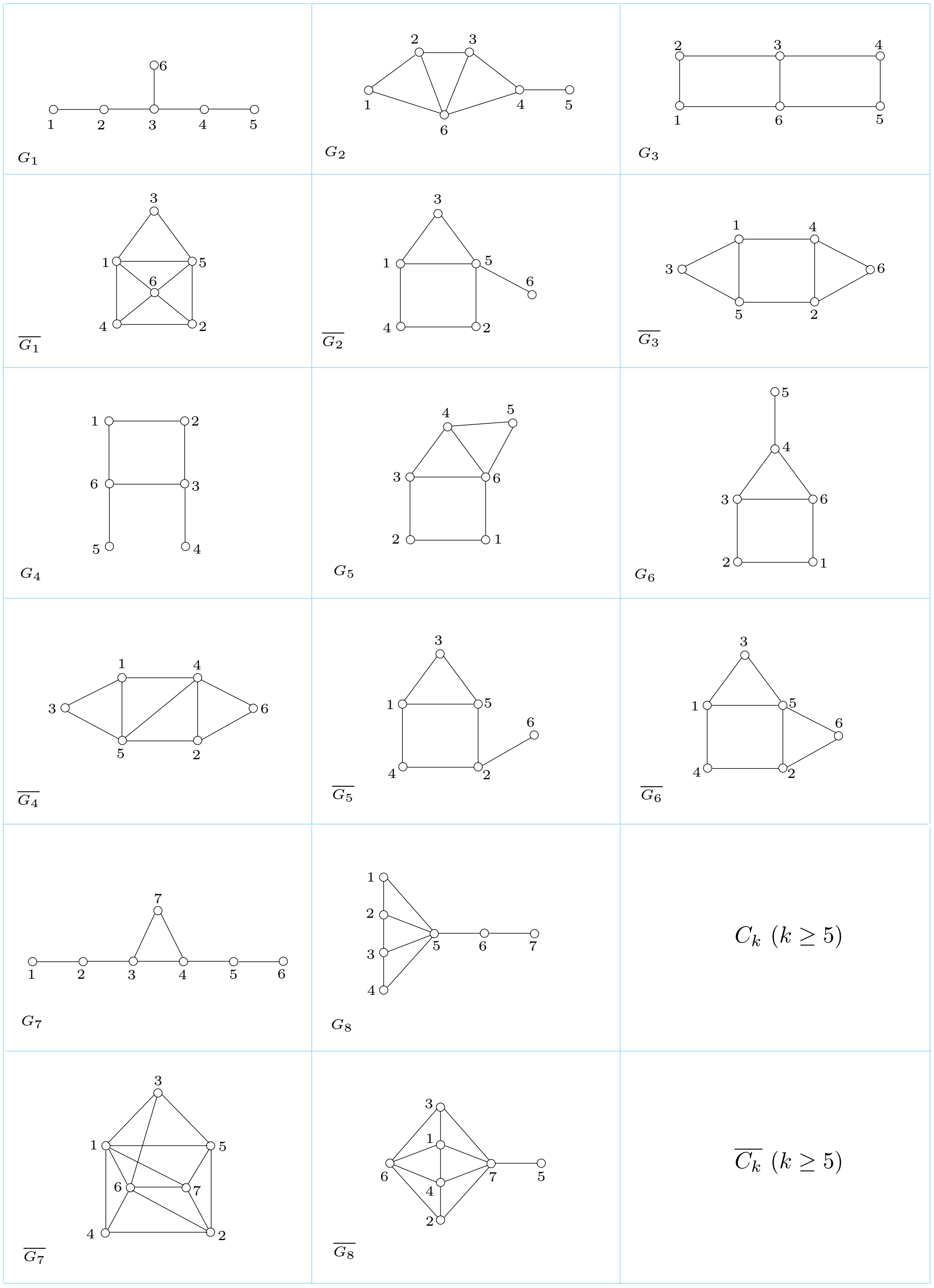

FIG. 3.1. *Forbidden induced subgraphs.*

Note that G is not split-perfect if and only if $\overline{G}$ is not split-perfect. Thus we only show that none of C_k, $k \geq 5$, and G_i, $1 \leq i \leq 8$, is split-perfect.

Consider C_k for odd $k \geq 5$. In this case, all vertices of the C_k are pairwise equivalent, which means that C_k is not split-perfect. (Note that for odd cycles C_k, $k \geq 5$, it also follows from Reed's theorem that they are not split-perfect because they are not perfect.)

Let $k = 2n \geq 6$ and write $C_k = v_1v_2 \dots v_{2n}$. In this case, all odd vertices v_{2i-1} are pairwise equivalent and all even vertices v_{2i} are pairwise equivalent. Thus, if C_{2n} is split-perfect and H is a corresponding split graph, then, by balance, one class of H consists of exactly the vertices v_{2i-1} and the other class consists of exactly the vertices v_{2i}. Now it is a matter of routine to check that in any realization of the split graph H some P_4 in C_{2n} must be bad.

Assume that $G \in \{G_1, G_3, G_4\}$ is split-perfect and let H be a corresponding split graph. Then $2 \sim 4 \sim 6$. Since the P_4's in G are balanced, the classes of H are $\{2, 4, 6\}$ and $\{1, 3, 5\}$. Again, it is a matter of routine to check that in any realization of the split graph H some P_4 in G must be bad.

Similary, assume that $G \in \{G_2, G_6\}$ is split-perfect and let H be a corresponding split graph. Then $1 \sim 2 \sim 5$. By balance, the classes of H are $\{1, 2, 5\}$ and $\{3, 4, 6\}$. Again, it is a matter of routine to check that in any realization of the split graph H some P_4 in G must be bad.

If G_5 is split-perfect, then 1, 3, 4, 5, and 6 are pairwise equivalent. But then no P_4 in G_5 is balanced.

If G_7 is split-perfect, then $3 \sim 4 \sim 7$. Since every P_4 of G_7 has two vertices in $\{3, 4, 7\}$, it follows by balance that every corresponding split graph H has classes $\{3, 4, 7\}$ and $\{1, 2, 5, 6\}$. Again, it is a matter of routine to check that in any realization of the split graph H some P_4 in G_7 must be bad.

Finally, if G_8 is split-perfect, then 1, 2, 3, and 4 are pairwise equivalent, but induce a P_4. ◻

4. Double-split graphs. We define now the class of double-split graphs generalizing the split graphs and playing a key role in the subsequent characterization of split-perfect graphs. As an important step towards this characterization, we will show that double-split graphs are split-perfect.

DEFINITION 4.1. *A graph is called* double-split *if it can be obtained from two disjoint (possibly empty) split graphs* $G_L = (Q_L, S_L, E_L)$, $G_R = (Q_R, S_R, E_R)$ *and an induced path* $P = P[x_L, x_R]$*, possibly empty, by adding all edges between* x_L *and vertices in* Q_L *and all edges between* x_R *and vertices in* Q_R *(see Figure* 4.1*).*

Remark. Every split graph is double-split as the case of an empty path P and an empty split graph G_R shows.

LEMMA 4.2. *Double-split graphs are split-perfect.*

Proof. Let G be a double-split graph consisting of two split graphs $G_L = (Q_L, S_L, E_L)$, $G_R = (Q_R, S_R, E_R)$ with cliques Q_L, Q_R and stable sets S_L, S_R. If the path P connecting G_L and G_R is empty, then G is P_4-isomorphic to the following split graph $H = (Q_L \cup Q_R, S_L \cup S_R, E_H)$ obtained from G_L and G_R by adding a join between Q_L and Q_R and between S_L and Q_R.

Now assume that $P = v_3v_4 \dots v_i$, $i \geq 3$, such that $x_L = v_3$ is adjacent to all vertices of Q_L and $x_R = v_i$ is adjacent to all vertices of Q_R. We construct a split graph $H = (Q_H, S_H, E_H)$ with the same P_4-structure as G. Hereby we use the fact that induced paths $P' = v_1v_2v_3v_4 \dots v_iv_{i+1}v_{i+2}$ are split-perfect and can be realized by the elementary split graph $G_{P'} = (\{v_2, v_4, v_6, \dots\}, \{v_1, v_3, v_5, \dots\}, E_{P'})$. We will

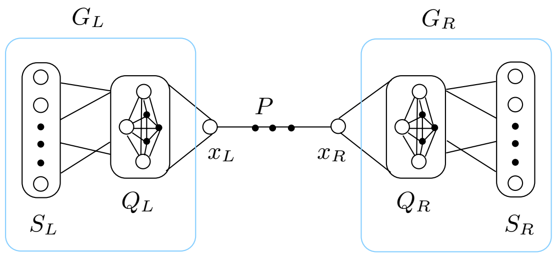

FIG. 4.1. *Double-split graphs illustrated.*

see that this split graph $G_{P'}$ can be extended to H by replacing v_1 by S_L, v_2 by Q_L, v_{i+1} by Q_R, and v_{i+2} by S_R in a suitable way. Moreover, we use the following simple property of split graphs.

CLAIM. *Let $G = (Q, S, E)$ be a split graph and let $G' = (Q, S, E')$ be the following* bipartite complement *of G: For all $x \in Q$ and all $y \in S$, $xy \in E' \iff xy \notin E$. Then G and G' are P_4-isomorphic.* □

We construct the split graph $H = (Q_H, S_H, E_H)$ depending on the parity of $|P|$; see Figure 4.2.

$$Q_H := \begin{cases} Q_L \cup \{v_4, v_6, \ldots, v_{i-1}\} \cup Q_R & \text{if } i \text{ is odd,} \\ Q_L \cup \{v_4, v_6, \ldots, v_i\} \cup S_R & \text{otherwise,} \end{cases}$$

$$S_H := \begin{cases} S_L \cup \{v_3, v_5, \ldots, v_i\} \cup S_R & \text{if } i \text{ is odd,} \\ S_L \cup \{v_3, v_5, \ldots, v_{i-1}\} \cup Q_R & \text{otherwise.} \end{cases}$$

Now E_H consists of the following edges based on the edge set of $G_{P'}$ and on E_L, E_R and depending on the parity of $|P|$:

(1) vertices in Q_H are pairwise adjacent;
(2) the E_H-edge set between S_L and Q_L is E_L;
(3) the E_H-edge set between Q_R and S_R is the bipartite complement of E_R if i is odd and is E_R otherwise;
(4) there is a join between Q_L and S_R (due to the fact that there is an edge between v_2 and v_{i+2} in $G_{P'}$) if i is odd and there is a join between Q_L and Q_R otherwise;
(5) vertices from $\{v_3, v_4, \ldots, v_i\}$ have a join to a set from S_L, Q_L, Q_R, S_R if and only if there is an edge in $G_{P'}$ to the corresponding vertex from $\{v_1, v_2, v_{i+1}, v_{i+2}\}$. Thus, for odd i, Q_L has a join to $v_5, v_7, \ldots, v_i$, all vertices $x \in \{v_4, v_6, \ldots, v_{i-1}\}$ have a join to S_R, and Q_R has a join to v_i; if i is even, then Q_L has a join to $v_5, v_7, \ldots, v_{i-1}$, and all vertices $x \in \{v_4, v_6, \ldots, v_{i-2}\}$ have a join to Q_R;
(6) the edges between vertices from $v_3, v_4, \ldots, v_i$ are the same as in $G_{P'}$.

We claim that G and H are P_4-isomorphic. First we show that every P_4 of G is a P_4 in H. There are the following types of P_4's in G:

(a) P_4's in G_L and P_4's in G_R;
(b) xyv_3v_4 with $x \in S_L$, $y \in Q_L$, $xy \in E_L$ (for $i = 3$ replace v_4 by a vertex $z \in Q_R$);
(c) $xv_3v_4v_5$ for $x \in Q_L$ (for $i = 3$ replace v_4 by a vertex $y \in Q_R$ and v_5 by a vertex $z \in S_R$);

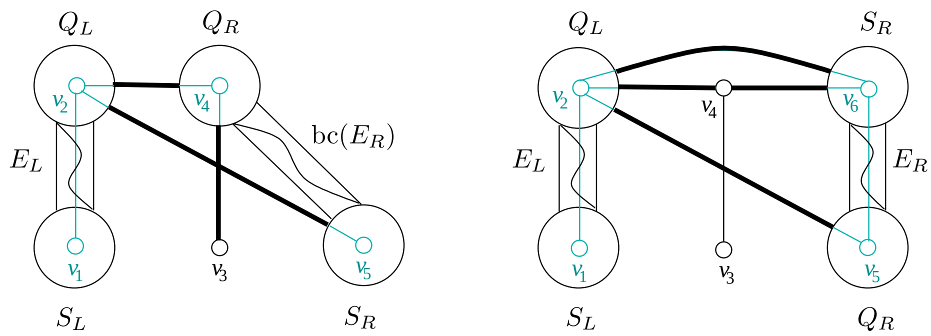

FIG. 4.2. *Construction for* $i = 3$ *(left) and* $i = 4$ *(right);* $\mathrm{bc}(E)$ *means the bipartite complement of* E*.*

(d) P_4's in $v_3, v_4, \ldots, v_i$ (for $i \in \{3,4,5\}$ there are no such P_4's);

(e) $v_{i-2}v_{i-1}v_ix$ with $x \in Q_R$ (for $i = 3$ this corresponds to case (b), for $i = 4$ replace v_{i-2} by $z \in Q_L$);

(f) $v_{i-1}v_ixy$ with $x \in Q_R$, $y \in S_R$, $xy \in E_R$ (for $i = 3$ replace v_{i-1} by $z \in Q_L$).

Type (a) for G_L is obviously fulfilled by construction of H, and for G_R, the bipartite complement of G_R in H ensures the property if i is odd, and is obvious in the other case.

Types (b), (c), (d), (e) are obviously fulfilled.

Type (f): For the P_4 $v_{i-1}v_ixy$ with $x \in Q_R$, $y \in S_R$, $xy \in E_R$, if i is odd, then xy is not an edge in the bipartite complement of G_R, and thus $v_ixv_{i-1}y$ is a P_4 in H. If i is even, xy is an edge in E_H and $v_{i-1}v_iyx$ is a P_4 in H.

Now consider a P_4 in H. According to the definition of H this is either a P_4 between Q_L and S_L which is the same as in G_L, or a P_4 between Q_R and S_R which, for odd i, is the same as in G_R due to the bipartite complement and, for even i, is obviously the same as in G_R, or a P_4 which goes back to $G_{P'}$ but $G_{P'}$ realizes exactly the P_4's of the induced path P' which are P_4's in G as well. □

Double-split graphs and their complements can be recognized in linear time due to their simple structure as we will show in the appendix.

5. The structure of split-perfect graphs. Now we are able to describe prime split-perfect graphs as follows.

THEOREM 5.1. *Let* G *be a prime graph. Then the following statements are equivalent:*

(i) G *is split-perfect;*

(ii) G *has no induced subgraphs* $C_k, \overline{C_k}$ $(k \geq 5)$, $G_i, \overline{G_i}$ $(1 \leq i \leq 8)$*;*

(iii) G *or* $\overline{G}$ *is a double-split graph.*

Theorem 5.1 and Propositions 2.3 and 2.6 immediately yield the following theorem.

THEOREM 5.2. *A graph* G *is split-perfect if and only if each of its* p*-connected components* H *has the following properties: Every homogeneous set in* H *induces a* P_4*-free graph, and* H^* *is a double-split graph or the complement of a double-split graph.* □

Proof of Theorem 5.1. The implication (i) ⇒ (ii) follows from Lemma 3.1, and the implication (iii) ⇒ (i) follows from Lemma 4.2. Note that these two implications hold in general, not only for p-connected graphs or prime graphs.

We now complete the proof by showing (ii) ⇒ (iii), where we will make use of the primality as follows.

OBSERVATION 5.3. *Let G be prime and let H be a P_4-free induced subgraph of G. If H is not a stable set (a clique, respectively), then there exist adjacent (nonadjacent, respectively) vertices x, y in H and a vertex z outside H such that z is adjacent to x and nonadjacent to y.*

Proof. Assume that H is not a stable set (the case that H is not a clique can be seen similarly). Let $S \subseteq H$ be maximal such that $H[S]$ has no isolated vertices. As H is not a stable set, $|S| \geq 2$. It is well known that P_4-free graphs with at least two vertices contain two vertices u and v with $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$ (so-called *twins*). Let $\{u, v\}$ be twins in S. As G is prime, there is a vertex $z \notin S$ adjacent to u and nonadjacent to v. By definition of S, $z \notin H$. If u and v are adjacent, then we are done by setting $x = u$ and $y = v$. Thus, let u and v be nonadjacent. By definition of S, u is adjacent to another vertex w in S which is also adjacent to v because $\{u, v\}$ is homogeneous in S. Now, we are done by setting $x = u$, $y = w$ (if z is nonadjacent to w), or $x = w$, $y = v$ (otherwise). ∎

Let G be a prime graph satisfying the statement (ii). If G is $(P_5, \overline{P_5})$-free, then by Lemma 2.1 G cannot contain a C_4 or a $\overline{C_4}$ (otherwise G would contain a G_3, $\overline{G_3}$, G_4, or $\overline{G_4}$). Hence G is $(C_4, \overline{C_4}, C_5)$-free, i.e., G is a split graph and we get (iii).

Therefore, we may assume that G contains a P_5 or a $\overline{P_5}$. By considering complementation if necessary, assume that G has an induced P_5. Consider a longest induced path $P = v_1 v_2 \ldots v_k$ in G. By assumption, $k \geq 5$. Now we are going to show, by a number of claims, that G is a double-split graph.

CLAIM NO-MIDDLE. *For every $2 < i < k - 1$,*

$$(N(v_{i-1}) \cap N(v_{i+1})) - (N(v_{i-2}) \cup N(v_{i+2})) = \{v_i\}.$$

Proof. Let $H = (N(v_{i-1}) \cap N(v_{i+1})) - (N(v_{i-2}) \cup N(v_{i+2}))$. Then H induces a P_4-free graph, otherwise G would have a $\overline{G_8}$. Thus, assuming $H \neq \{v_i\}$, H has twins $\{x, y\}$. As G has no homogeneous set, there is a vertex $z \notin H$ such that $zx \in E(G)$ but $zy \notin E(G)$. We distinguish between three cases.

Case 1. z is adjacent to both v_{i-1} and v_{i+1}.

By definition of H and $z \notin H$, z must be adjacent to v_{i-2} or v_{i+2}. By symmetry, let $zv_{i-2} \in E(G)$. Now, if z is also adjacent to v_{i+2}, then $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}, y, z$ induce a $\overline{G_6}$. If z is nonadjacent to v_{i+2}, then the same vertices induce a $\overline{G_5}$. Case 1 is settled.

Case 2. z is adjacent to v_{i-1} and nonadjacent to v_{i+1} (or vice versa).

Then z cannot be adjacent to v_{i+2} (otherwise there is a C_5). Now, if x and y are adjacent, then there is a G_2, and if x, y are nonadjacent, then there is a $\overline{G_5}$. Case 2 is settled.

Case 3. z is nonadjacent to both v_{i-1} and v_{i+1}.

First, assume $xy \in E(G)$. Then z cannot be adjacent to v_{i-2} or to v_{i+2} (otherwise there is a G_5). But then $v_{i-2}, v_{i-1}, x, v_{i+1}, v_{i+2}, z$ induce a G_1. Second, assume $xy \notin E(G)$. Then there is a G_3 (if z is adjacent to v_{i-2}) or a G_4 (otherwise). Case 3 is settled. ∎

Let M be the set of all vertices outside P adjacent to a vertex in P but not to all vertices in P.

CLAIM N. *For every $v \in M$, $N(v) \cap P = \{v_2\}$ or $\{v_2, v_3\}$ or $\{v_1, v_2, v_3\}$ or $\{v_{k-1}\}$ or $\{v_{k-2}, v_{k-1}\}$ or $\{v_{k-2}, v_{k-1}, v_k\}$.*

Proof. Since G does not have a C_ℓ ($\ell \geq 5$), G_2, $\overline{G_2}$, G_3, G_5, or $\overline{G_6}$, every vertex in M has at most three neighbors in P. We distinguish between three cases.

Case 1. $|N(v) \cap P| = 3$.

Then $N(v) \cap P$ is a subpath of P, otherwise G would have a C_ℓ for some $\ell \geq 5$, or a G_3 or $\overline{G_5}$ or a G_6. Thus $N(v) \cap P = \{v_{i-1}, v_i, v_{i+1}\}$ for some suitable i. Now, by Claim No-Middle, $i = 2$ or $i = k-1$, and Case 1 is settled.

Case 2. $|N(v) \cap P| = 2$.

We first claim that $N(v) \cap P$ is a subpath of P. Assume to the contrary that the two neighbors of v in P are nonadjacent. Then there is a suitable i such that $N(v) \cap P = \{v_{i-1}, v_{i+1}\}$, otherwise G would have a C_ℓ for some $\ell \geq 5$. Now, by Claim No-Middle, $i = 2$ or $i = k-1$. By symmetry we only consider the case $N(v) \cap P = \{v_1, v_3\}$.

Let $H = (N(v_1) \cap N(v_3) \cap M) \cup \{v_2\}$. Note that no vertex in H is adjacent to a v_j, $j \geq 4$ (as we have seen in Case 1). Thus H is P_4-free (otherwise G would have a G_8). Since H is not a clique (it contains v and v_2), there exist, by Observation 5.3, nonadjacent vertices $x, y \in H$ and a vertex $z \notin H$ adjacent to x but nonadjacent to y. Note that $z \in M$: If z is nonadjacent to P, then v_1, v_3, x, y, z, v_4 induce a G_4. If z is adjacent to all v_i's, then v_1, v_3, v_4, v_5, y, z induce a G_5. Now, if $zv_3 \in E(G)$, then $zv_1 \notin E(G)$ (otherwise $z \in H$). But then v_1, v_3, v_4, x, y, z induce a $\overline{G_2}$ or a G_5. Thus, $zv_3 \notin E(G)$. But then v_1, v_3, v_4, x, y, z induce a G_3 or contain a C_5 depending on $zv_1 \in E$ (if $zv_4 \in E(G)$) or a G_4 or a $\overline{G_5}$ (if $zv_4 \notin E(G)$).

We have shown that the two neighbors of v on P are v_i and v_{i+1} for some suitable i. Since G has no G_7, $i \in \{1, 2, k-1, k-2\}$. We are going to show that $i \in \{2, k-2\}$ holds. By symmetry, we only show $i \neq 1$.

Assume to the contrary that $i = 1$. Let $H = N(v_2) - N(v_3)$. Then no vertex in H is adjacent to v_j, $j \geq 4$ (as we have seen in Case 1). Thus, H is P_4-free (otherwise G would have a G_8). Since H is not a stable set (it contains v and v_1), there exist, by Observation 5.3, adjacent vertices $x, y \in H$ and vertex $z \notin H$ adjacent to x but nonadjacent to y. If $zv_2 \in E(G)$, then $zv_3 \in E(G)$ (otherwise $z \in H$) and v_2, v_3, v_4, x, y, z induce a G_2 or a $\overline{G_4}$. Thus $zv_2 \notin E(G)$, hence also $zv_3 \notin E(G)$ (otherwise v_2, v_3, v_4, x, y, z induce a $\overline{G_5}$ or a $\overline{G_3}$). But then $zxv_2v_3 \cdots v_k$ is an induced path longer than P, or else z is adjacent to some v_j, $j \geq 4$, yielding a C_{j+1}.

This shows that $i \neq 1$ and, by symmetry, $i \neq k-1$. We have proved Claim N in Case 2.

Case 3. $|N(v) \cap P| = 1$.

Then $N(v) \cap P = \{v_2\}$ or $N(v) \cap P = \{v_{k-1}\}$. Otherwise G would have a G_1, or there is an induced path longer than P. Claim N is proved in Case 3. □

Let $Q_L = N(v_3) - (N(v_4) \cup N(v_5))$.

CLAIM QL. *Q_L is a clique.*

Proof. First note that Q_L induces a P_4-free graph (otherwise G would have a G_8). Now, assume to the contrary that Q_L is not a clique. By Observation 5.3, there exist nonadjacent vertices $x, y \in Q_L$ and vertex $z \notin Q_L$ adjacent to x but nonadjacent to y. We distinguish between two cases.

Case 1. z and v_3 are nonadjacent.

If $zv_4 \notin E(G)$, then x, y, z, v_3, v_4, v_5 induce a G_1 if $zv_5 \notin E$, else z, x, v_3, v_4, v_5 is a C_5. If $zv_4 \in E(G)$, then, by Claim N, $zv_5 \notin E(G)$ and G has a G_4. Case 1 is settled.

Case 2. z and v_3 are adjacent.

Then $z \in M$. Because, if z is adjacent to all v_i's, then y cannot be adjacent to v_2 (otherwise G would have a $\overline{G_4}$ induced by y, v_2, v_3, v_4, v_5, and z). By Claim N, y is also nonadjacent to v_1. But then G has a G_1.

Now, by definition of Q_L, z must be adjacent to v_4 or v_5, and by Claim N, z is adjacent to v_4 and nonadjacent to v_2. Thus x cannot be adjacent to v_2, otherwise

v_2, v_3, v_4, v_5, x, z induce a G_2 (if $zv_5 \notin E(G)$) or a $\overline{G_4}$ (if $zv_5 \in E(G)$). Therefore, by Claim N, x cannot be adjacent to v_1. But then $v_1, v_2, v_3, v_4, v_5, x$ induce a G_1. Case 2 is settled. ∎

Let T be the set of all vertices that are adjacent to all vertices in P, and let $S_L = N(Q_L) - (\{v_3\} \cup T)$.

CLAIM SL. *S_L is a stable set.*

Proof. We first show that

$$v \in S_L \Longrightarrow vv_i \notin E(G), \qquad i \geq 3. \tag{5.1}$$

Proof of (5.1). Assume first that v is adjacent to v_3. By definition of Q_L, v must be adjacent to v_4 or to v_5 (otherwise v would belong to Q_L, contradicting $v \in S_L$). Thus, by Claim N, v is adjacent to v_4 and is nonadjacent to v_1, v_2. Now, a neighbor x in Q_L of v together with v_1, v_2, v_3, v_4, and v induce a G_2 (if $xv_1 \notin E(G)$) or a $\overline{G_4}$ (otherwise). We have shown that v is nonadjacent to v_3. Next, if vv_4 is an edge, then, by Claim N, v is nonadjacent to v_1 and v_2, and so a neighbor x in Q_L together with v_1, v_2, v_3, v_4, v induce a G_6 or a G_5. Thus v is nonadjacent to v_4. Finally, v cannot be adjacent to v_i for any $i \geq 5$ because G does not have a C_ℓ, $\ell \geq 5$. Thus, (5.1) is proved. ∎

Next, we show that

$$\text{for every two adjacent vertices } u, v \in S_L,\ N(u) \cap Q_L = N(v) \cap Q_L. \tag{5.2}$$

Proof of (5.2). Assume that there is a vertex $x \in Q_L$ adjacent to u but nonadjacent to v, say. Let $y \in Q_L$ be a neighbor of v. Then by (5.1), u, v, x, y, v_3, v_4 induce a G_2 (if $yu \in E(G)$) or a G_6 (otherwise). This contradiction proves (5.2). ∎

We furthermore show that

$$S_L \text{ induces a } P_4\text{-free graph.} \tag{5.3}$$

Proof of (5.3). If not, then by (5.2), there is a vertex in Q_L adjacent to all vertices of a P_4 in S_L. By (5.2), G would have a G_8. This proves (5.3). ∎

Now, to finish the proof of Claim SL, assume that S_L is not a stable set. By Observation 5.3, there exist adjacent vertices $u, v \in S_L$ and vertex $w \notin S_L$ adjacent to u but nonadjacent to v. By (5.2), $w \notin Q_L$.

Since $w \notin S_L$, w cannot have a neighbor in Q_L, and it can be seen, as in the proof of (5.1), that w cannot be adjacent to v_i, $i \geq 3$. But then $wuxv_3v_4 \cdots v_k$, where $x \in Q_L$ is a neighbor of u, is an induced path longer than P. The proof of Claim SL is complete. ∎

Let $Q_R = N(v_{k-2}) - (N(v_{k-3}) \cup N(v_{k-4}))$ and $S_R = N(Q_R) - (\{v_{k-2}\} \cup T)$. By symmetry, we have the following claims.

CLAIM QR. *Q_R is a clique.* ∎

CLAIM SR. *S_R is a stable set.* ∎

Note that from the definition it follows that $Q_L \cap Q_R = \emptyset$, and from Claim N and the forbidden $\overline{G_6}$ it follows that $S_L \cap S_R = \emptyset$.

CLAIM NOE (no other edge). *There is no edge between $Q_L \cup S_L$ and $Q_R \cup S_R$.*

Proof. Let $x \in Q_L \cup S_L$ and $y \in Q_R \cup S_R$ be two adjacent vertices. Since P is an induced path and by Claim N, $x, y \notin \{v_1, v_2, v_{k-1}, v_k\}$. Then $x \notin Q_L$ (otherwise y would belong to S_L) and $y \notin Q_R$ (otherwise x would belong to S_R). Thus, $x \in S_L$ and $y \in S_R$, yielding a C_k, $k \geq 5$. This contradiction proves Claim NOE. ∎

CLAIM NOV (no other vertex). *$V(G) = P \cup M \cup S_L \cup S_R \cup T$.*

Proof. If there is a vertex $v \notin P \cup M \cup S_L \cup S_R \cup T$, then, as G is connected (it has no homogeneous set), v must be adjacent to some vertex in $S_L \cup S_R$. But then there is an induced path longer than P. ◻

CLAIM T. *$T = \emptyset$, i.e., there is no vertex adjacent to all vertices of P.*

Proof. Assume there is a vertex v adjacent to all v_i's. Then v is adjacent to all vertices in Q_L (and in Q_R), otherwise G would have a $\overline{G_4}$. Also, v is adjacent to all vertices in S_L (and in S_R), otherwise G would have a G_2.

Thus, every vertex from T is adjacent to all vertices in $G-T$, implying, by Claim NOV, that $G-T$ is a homogeneous set in G. This contradiction proves Claim T. ◻

It follows from the claims that G is a double-split graph (with the two split graphs formed by Q_L, S_L and Q_R, S_R, respectively). The proof of Theorem 5.1 is complete. ◻

COROLLARY 5.4. *Split-perfect graphs can be recognized in linear time.*

Proof. This follows from Theorem 5.2 and the facts that

- the p-connected components of a graph can be found in linear time [6];
- all maximal homogeneous sets of a (p-connected) graph G can be found in linear time [23, 24, 45];
- P_4-free graphs can be recognized in linear time [21] (for a new and simpler 3-sweep lexicographic breadth-first search algorithm recognizing P_4-free graphs in linear time, see [13]); and
- double-split graphs and their complements can be recognized in linear time (see the appendix). ◻

In the remainder of this section we will show that the class of split-perfect graphs lies between the classes of superbrittle graphs and of brittle graphs. We first give a new characterization of superbrittle graphs in the following theorem.

THEOREM 5.5. *A graph G is superbrittle if and only if for each of its p-connected components H of G,*

(i) *the homogeneous sets of H are cographs, and*

(ii) *the characteristic graph H^* is a split graph.*

Proof. Assume first that G is superbrittle. Then, since the graphs G_8 and $\overline{G_8}$ (see Figure 3.1) are not superbrittle, homogeneous sets in p-connected components are P_4-free; otherwise a crossing P_4 leads to an induced subgraph G_8 or $\overline{G_8}$. Now we show condition (ii). Note first that obviously superbrittle graphs are also $(P_5, \overline{P_5}, C_5, G_4, \overline{G_4})$-free (for G_4 and $\overline{G_4}$, see Figure 3.1). Then, due to Lemma 2.1, H^* is C_4-free since a C_4 in a characteristic graph extends into a $\overline{P_5}$ or G_3 or G_4 but the G_3 contains a P_5. The same holds for the complements which means that H^* and its complement are chordal, i.e., H^* is a split graph.

Now let G be a graph fulfilling the conditions (i) and (ii) for all its p-connected components. We are going to show that G is superbrittle. Since the property to be superbrittle is a P_4 condition, it is sufficient to show that the p-connected components H of G are superbrittle. Note that split graphs are superbrittle, i.e., H^* is superbrittle. Furthermore, by substituting cographs as homogeneous sets into vertices of a split graph, no midpoint of a P_4 in H^* can become an endpoint in H and no endpoint of a P_4 in H^* can become a midpoint in H since homogeneous sets contain at most one vertex of a P_4. This shows that H is superbrittle, and thus G is superbrittle. ◻

Theorem 5.5 immediately implies the following.

COROLLARY 5.6. *Superbrittle graphs are split-perfect and can be recognized in linear time.*

COROLLARY 5.7. *Split-perfect graphs are brittle. Moreover, a perfect order of a split-perfect graph can be constructed efficiently.*

Proof. Since there is no crossing P_4 for two p-connected components, a graph is brittle if and only if each of its p-connected components is brittle. Now, if G is a p-connected split-perfect graph, then G^* is chordal or the complement of a chordal graph (Theorem 5.2); hence G^* is brittle. Let v be a vertex in G^* that is not an endpoint (a midpoint) of any P_4 in G^*. Then, by Proposition 2.5, every vertex in the homogeneous set in G corresponding to v is not an endpoint (a midpoint, respectively) of any P_4 in G. Since every induced subgraph of a split-perfect graph is again split-perfect, it follows that split-perfect graphs are brittle.

Moreover, a perfect order of a split-perfect graph can be constructed as follows: Note that a perfect order of a chordal graph (the complement of a chordal graph) can be found by constructing a perfect elimination order and reversing its order. Now, a perfect order of G^* yields, in a natural way, a perfect order of G. Combining these perfect orders on the p-connected components in an arbitrary sequence, we obtain a perfect order of a split-perfect graph. □

6. Optimization in split-perfect graphs. As already mentioned, Theorem 2.2 implies a decomposition scheme, called *primeval decomposition*, for arbitrary graphs. The corresponding tree representation, called *primeval tree*, has the p-connected components and vertices not belonging to any P_4 of the considered graph as its leaves.

The important features of the primeval tree of a given graph G are the following:

- If an optimization problem such as weighted clique number, weighted chromatic number, weighted independence number, and weighted clique cover number can be solved efficiently on the p-connected components of G, then one can also efficiently solve the problem on the whole graph G; see, for example, [1].
- The primeval tree can be constructed in linear time; see [6].

Based on these facts, linear time or at least polynomial time algorithms have been found for classical NP-hard problems on many graph classes such as $(q, q-4)$-graphs and various subclasses. We now point out how to compute the weighted clique size $\omega_{\mathrm{w}}(G)$ and the weighted independence number $\alpha_{\mathrm{w}}(G)$ for p-connected split-perfect graphs G efficiently.

First, we shall use the following facts:

- The weighted clique number of a chordal graph can be computed in linear time (well known).
- The weighted independence number of a chordal graph can be computed in linear time as pointed out by Frank [27].

Second, let H be a homogeneous set in G and let G/H be the graph obtained from G by contracting H to a single vertex v_H. Then it is well known (and easy to see) that

$$\omega_{\mathrm{w}'}(G/H) = \omega_{\mathrm{w}}(G), \quad \text{respectively}, \quad \alpha_{\mathrm{w}'}(G/H) = \alpha_{\mathrm{w}}(G),$$

where the weighting w' is obtained from w by defining $\mathrm{w}'(v_H) = \omega_{\mathrm{w}}(G[H])$, respectively, $\mathrm{w}'(v_H) = \alpha_{\mathrm{w}}(G[H])$.

Thus, if $\omega_{\mathrm{w}}(G^*)$ and $\omega_{\mathrm{w}}(H)$ (respectively, $\alpha_{\mathrm{w}}(G^*)$ and $\alpha_{\mathrm{w}}(H)$), H a homogeneous set in G, can be computed in linear time, then $\omega_{\mathrm{w}}(G)$ (respectively, $\alpha_{\mathrm{w}}(G)$) can be computed in linear time, too.

Now, if G is a p-connected split-perfect graph, then by Theorem 5.1, G^* is a double-split graph or the complement of a double-split graph. In any case, G^* is a chordal graph or the complement of a chordal graph. If G is chordal, then $\omega_{\mathrm{w}}(G^*)$

and $\alpha_w(G^*)$ can be computed in linear time. If G^* is the complement of a chordal graph, then, by considering $\overline{G^*}$, $\omega_w(G^*)$ and $\alpha_w(G^*)$ can be computed in $O(n^2)$ time (n is the vertex number of G). Furthermore, by Proposition 2.5, every homogeneous set H of G induces a P_4-free graph; hence $\omega_w(H)$ and $\alpha_w(H)$ can be computed in linear time. This and the facts that the primeval tree of G as well as all maximal homogeneous sets of G can be found in linear time show that $\omega_w(G)$ and $\alpha_w(G)$ can be computed in $O(n^2)$ time.

The problems of weighted chromatic number and weighted clique cover number can be solved similarly; we omit the details. Note that for perfect graphs in general and in particular for split-perfect graphs, the weighted chromatic number equals the weighted clique number, and the weighted independence number equals the weighted clique cover number. Thus, we can state the following result.

THEOREM 6.1. *The weighted clique number, the weighted chromatic number, the weighted independence number, and the weighted clique cover number of a split-perfect graph can be computed in $O(n^2)$ time.*

Appendix. Linear-time recognition of double-split graphs and their complements. Let $DS(k)$ denote the class of double-split graphs (H_1, P, H_2) with split graphs H_1 and H_2 and k vertices in the induced path P connecting H_1 with H_2, and let $DS = \bigcup_{k\geq 1} DS(k)$.

THEOREM A.1. *Double-split graphs and their complements can be recognized in linear time.*

Proof. For a given graph $G = (V, E)$ we have to check whether there is a $k \geq 1$ such that $G \in DS(k)$. Observe that for $G = (H_1, P, H_2) \in DS(k)$ with $k \geq 3$, the path $P = x_1 \dots x_k$ contains at least one inner vertex of degree 2.

Thus, in order to check whether $G \in DS(k)$ for $k \geq 3$, determine the set D_2 of vertices of degree 2 in G (in the nondegenerate case, D_2 contains no clique vertices from H_1, H_2 and thus D_2 is stable) and check whether $G \setminus D_2$ is the disjoint union of two split graphs H'_1, H'_2. Moreover, check whether D_2 is the disjoint union of an induced path P' (the inner vertices of P) and a stable set S'. S'_i consists of the vertices in S' adjacent to some vertex in H'_i for $i \in \{1, 2\}$ (i.e., $H'_i \cup S'_i$ is a split graph H_i with the property that the left (right) endvertex of P' is adjacent to exactly one clique vertex of H_1 (H_2, respectively)).

Now consider the case $G \in DS(1)$ or $G \in DS(2)$. We give an argument using P_4 properties that is similar for the complement graphs.

Case ($G \in DS(1)$). For a given G we have to identify the vertex x_1 of P. If $G \in DS(1)$, G has the following two types of P_4's:

(1) P_4's $abcd$ contained in H_1 (H_2, respectively);

(2) P_4's abx_1d containing x_1 as a midpoint.

Thus for a given G, find a P_4 in linear time if there is any (the case that G contains no P_4 reduces to threshold graphs or two cliques intersecting in exactly one vertex), and check whether one of the midpoints of the P_4 (of type (2)) is a cutpoint of G such that the connected components are split graphs and the midpoint is completely adjacent to both of the cliques. If none of the midpoints is a cutpoint, then check the P_4 $abcd$ (of type (1)) for the following property: Let $N := N(b) \cap N(c) \cap \overline{N}(a) \cap \overline{N}(d)$, where $\overline{N}(v)$ is the set of all nonneighbors of v. Check whether the two nontrivial connected components of $G' := G \setminus N$ are split graphs. If yes, then one of these split graphs (namely the one not containing the P_4) must have exactly one neighbor x_1 in N. Now check whether the neighborhoods of x_1 in the two components H_1, H_2 of $G \setminus \{x_1\}$ are cliques C_1, C_2 such that $H_i \setminus C_i$ are stable.

Case ($G \in DS(2)$). For a given G we have to identify the vertices x_1, x_2 of P. If $G \in DS(2)$, G has the following three types of P_4's:

(1) P_4's $abcd$ contained in H_1 (H_2, respectively);

(2) P_4's abx_1x_2 containing x_1 as a midpoint and x_2 as an endpoint;

(3) P_4's ax_1x_2b containing x_1, x_2 as midpoints.

Again we start with determining any P_4 in G. For types (2) and (3), try determining whether the midpoints of the P_4 are cutpoints and the connected components fulfill the required properties. For type (1), similar arguments as in case $G \in DS(1)$ will work.

Let co-$DS(k)$ denote the complement graphs of $DS(k)$ graphs. We first describe linear time recognition of co-$DS(k)$ graphs for $k \geq 3$. As for $DS(k)$ graphs, the inner vertices of the path P have to fulfill a degree condition which is now degree $n-3$. Thus, in order to check whether $G \in$ co-$DS(k)$ for $k \geq 3$, determine the set D_{n-3} of vertices of degree $n-3$ in G and check whether $G \setminus D_{n-3}$ is the join of two split graphs H'_1, H'_2. In order to check this in linear time, use the techniques of [24] in order to determine the (two nontrivial) connected components H'_1, H'_2 in the complement graph $\overline{G}$ for a given G and check whether they are split graphs. Moreover, check whether the connected components of D_{n-3} in the complement graph are an induced path P' (the inner vertices of P) and two sets S'_1, S'_2 such that $H'_i \cup S'_i$ is a split graph for $i \in \{1, 2\}$ with the property that the left (right) endvertex of P' is nonadjacent to exactly one clique vertex of H_1 (H_2, respectively).

Now consider the case $G \in$ co-$DS(1)$ or $G \in$ co-$DS(2)$. In these cases, using P_4 properties, we find the special vertex x_1 (special vertices x_1, x_2, respectively) as for $G \in DS(1)$ or $G \in DS(2)$, and using the techniques of [24], we find the connected components of $\overline{G}$ in linear time on input G. ◻

Acknowledgment. We are grateful to two anonymous referees for their careful reading and helpful comments.

REFERENCES

[1] L. Babel, *On the P_4-Structure of Graphs*, Habilitationsschrift, TU München, Munich, 1997.

[2] L. Babel, A. Brandstädt, and V. B. Le, *Recognizing the P_4-structure of bipartite graphs*, Discrete Appl. Math., 93 (1999), pp. 157–168.

[3] L. Babel, A. Brandstädt, and V. B. Le, *Recognizing the P_4-structure of claw-free graphs and of a larger class of graphs*, Discrete Math. Theor. Comput. Sci., 5 (2002), pp. 127–146; also available online from http://dmtcs.loria.fr/volumes/abstracts/dm050109.abs.html.

[4] L. Babel and S. Olariu, *On the structure of graphs with few P_4*, Discrete Appl. Math., 84 (1998), pp. 1–13.

[5] L. Babel and S. Olariu, *On the p-connectedness of graphs: A survey*, Discrete Appl. Math., 95 (1999), pp. 11–33.

[6] S. Baumann, *A Linear Algorithm for the Homogeneous Decomposition of Graphs*, Report M–9615, Zentrum Mathematik, TU München, Munich, 1996.

[7] C. Berge and V. Chvátal, eds., *Topics on Perfect Graphs*, Ann. Discrete Math. 21, North–Holland, Amsterdam, 1984.

[8] A. Brandstädt and V. B. Le, *Recognizing the P_4-structure of block graphs*, Discrete Appl. Math., 99 (2000), pp. 349–366.

[9] A. Brandstädt and V. B. Le, *Tree- and forest-perfect graphs*, Discrete Appl. Math., 95 (1999), pp. 141–162.

[10] A. Brandstädt, V. B. Le, and S. Olariu, *Linear-Time Recognition of the P_4-Structure of Trees*, Rutcor research report 19-96, Rutgers University, New Brunswick, NJ, 1996.

[11] A. Brandstädt, V. B. Le, and S. Olariu, *Efficiently recognizing the P_4-structure of trees and of bipartite graphs without short cycles*, Graphs Combin., 16 (2000), pp. 381–387.

[12] A. Brandstädt, V. B. Le, and J. Spinrad, *Graph Classes: A Survey*, SIAM Monogr. Discrete Math. Appl. 3, SIAM, Philadelphia, 1999.

[13] A. Bretscher, D. G. Corneil, M. Habib, and Ch. Paul, *A simple linear time LexBFS cograph recognition algorithm*, in Proceedings of the 29th Workshop on Graph-Theoretic Concepts in Computer Science (Elspeet, The Netherlands), Lecture Notes in Comput. Sci. 2880, Springer-Verlag, Berlin, 2003, pp. 119–130.
[14] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, *Cleaning for Bergeness*, manuscript, 2003.
[15] M. Chudnovsky and P. Seymour, *Recognizing Berge Graphs*, manuscript, 2003.
[16] V. Chvátal, *Perfect Graphs Seminar*, McGill University, Montreal, 1983.
[17] V. Chvátal, *Perfectly ordered graphs*, Ann. Discrete Math., 21 (1984), pp. 63–65.
[18] V. Chvátal, *A semi-strong perfect graph conjecture*, Ann. Discrete Math., 21 (1984), pp. 279–280.
[19] D. G. Corneil, H. Lerchs, and L. Stewart-Burlingham, *Complement reducible graphs*, Discrete Appl. Math., 3 (1981), pp. 163–174.
[20] D. G. Corneil, Y. Perl, and L. K. Stewart, *Cographs: Recognition, applications, and algorithms*, Congr. Numer., 43 (1984), pp. 249–258.
[21] D. G. Corneil, Y. Perl, and L. K. Stewart, *A linear recognition algorithm for cographs*, SIAM J. Comput., 14 (1985), pp. 926–934.
[22] G. Cornuéjols, X. Liu, and K. Vušković, *A Polynomial Algorithm for Recognizing Perfect Graphs*, manuscript, 2003.
[23] A. Cournier and M. Habib, *A new linear algorithm for modular decomposition*, in Trees in Algebra and Programming – CAAP '94, Lecture Notes in Comput. Sci. 787, Springer, Berlin, 1994, pp. 68–84.
[24] E. Dahlhaus, J. Gustedt, and R. M. McConnell, *Efficient and practical modular decomposition*, J. Algorithms, 41 (2001), pp. 360–387.
[25] G. Ding, *Recognizing the P_4-structure of a tree*, Graphs Combin., 10 (1994), pp. 323–328.
[26] S. Földes and P. L. Hammer, *Split graphs*, Congr. Numer., 19 (1977), pp. 311–315.
[27] A. Frank, *Some polynomial algorithms for certain graphs and hypergraphs*, Congr. Numer., 15 (1976), pp. 211–226.
[28] V. Giakoumakis, *P_4-laden graphs: A new class of brittle graphs*, Inform. Process. Lett., 60 (1996), pp. 29–36.
[29] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.
[30] P. L. Hammer and B. Simeone, *The splittance of a graph*, Combinatorica, 1 (1981), pp. 275–284.
[31] R. B. Hayward, S. Hougardy, and B. A. Reed, *Polynomial time recognition of P_4-structure*, in Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms, ACM, New York, SIAM, Philadelphia, 2002, pp. 382–389.
[32] C. T. Hoàng, *A Class of Perfect Graphs*, Ms. Sc. thesis, School of Computer Science, McGill University, Montreal, 1983.
[33] C. T. Hoàng, *Perfect Graphs*, Ph.D. thesis, McGill University, Montreal, 1985.
[34] C. T. Hoàng, *Perfectly orderable graphs: A survey*, in Perfect Graphs, Wiley-Intersci. Ser. Discrete Math. Optim., J. L. Ramirez-Alfonsin and B. A. Reed, eds., Wiley, Chichester, 2001, pp. 139–166.
[35] C. T. Hoàng and N. Khouzam, *On brittle graphs*, J. Graph Theory, 12 (1988), pp. 391–404.
[36] C. T. Hoàng and B. Reed, *Some classes of perfectly orderable graphs*, J. Graph Theory, 13 (1989), pp. 445–463.
[37] B. Jamison and S. Olariu, *P_4-reducible graphs – a class of uniquely tree representable graphs*, Stud. Appl. Math., 81 (1989), pp. 79–87.
[38] B. Jamison and S. Olariu, *A new class of brittle graphs*, Stud. Appl. Math., 81 (1989), pp. 89–92.
[39] B. Jamison and S. Olariu, *A unique tree representation for P_4-sparse graphs*, Discrete Appl. Math., 35 (1992), pp. 115–129.
[40] B. Jamison and S. Olariu, *A linear-time algorithm to recognize P_4-reducible graphs*, Theoret. Comput. Sci., 145 (1995), pp. 329–344.
[41] B. Jamison and S. Olariu, *Linear time optimization algorithms for P_4-sparse graphs*, Discrete Appl. Math., 61 (1995), pp. 155–175.
[42] B. Jamison and S. Olariu, *p-components and the homogeneous decomposition of graphs*, SIAM J. Discrete Math., 8 (1995), pp. 448–463.
[43] V. B. Le, *Bipartite-perfect graphs*, Discrete Appl. Math., 127 (2003), pp. 581–599.
[44] R. Lin and S. Olariu, *A fast parallel algorithm to recognize P_4-sparse graphs*, Discrete Appl. Math., 81 (1998), pp. 191–215.
[45] R. M. McConnell and J. Spinrad, *Modular decomposition and transitive orientation*, Discrete Math., 201 (1999), pp. 189–241.

[46] M. Middendorf and F. Pfeiffer, *On the complexity of recognizing perfectly orderable graphs*, Discrete Math., 80 (1990), pp. 327–333.

[47] M. Preissmann, D. de Werra, and N. V. R. Mahadev, *A note on superbrittle graphs*, Discrete Math., 61 (1986), pp. 259–267.

[48] B. Reed, *A semi-strong perfect graph theorem*, J. Combin. Theory Ser. B, 43 (1987), pp. 223–240.

[49] D. J. Rose, R. E. Tarjan, and G. S. Lueker, *Algorithmic aspects of vertex elimination on graphs*, SIAM J. Comput., 5 (1976), pp. 266–283.

[50] A. Schäffer, *Recognizing brittle graphs: Remarks on a paper of Hoàng and Khouzam*, Discrete Appl. Math., 31 (1991), pp. 29–35.

[51] J. P. Spinrad and J. L. Johnson, *Brittle, Bipolarizable, and P_4-Simplicial Graph Recognition*, manuscript, Department of Computer Science, Vanderbilt University, Nashville, TN, 1999.

[52] S. Sorg, *Die P_4-Struktur von Kantengraphen bipartiter Graphen*, Diploma thesis, Mathematisches Institut der Universität zu Köln, Cologne, 1997.