New Applications of Clique Separator Decomposition for the Maximum Weight Stable Set Problem

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Abstract. Graph decompositions such as decomposition by clique separators and modular decomposition are of crucial importance for designing efficient graph algorithms. Clique separators in graphs were used by Tarjan as a divide-and-conquer approach for solving various problems such as the Maximum Weight Stable Set (MWS) Problem, Coloring and Minimum Fill-in. The basic tool is a decomposition tree of the graph whose leaves have no clique separator (so-called *atoms*), and the problem can be solved efficiently on the graph if it is efficiently solvable on its atoms. We give new examples where the clique separator decomposition works well for the MWS problem which also improves and extends various recently published results. In particular, we describe the atom structure for some new classes of graphs whose atoms are P_5-free (the P_5 is the induced path with 5 vertices) and obtain new polynomial time results for MWS.

1 Introduction

In an undirected graph $G = (V, E)$, a *stable* (or *independent*) vertex set is a subset of mutually nonadjacent vertices. The *Maximum Weight Stable* (or *Independent*) *Set* (*MWS*) Problem asks for a stable set of maximum weight sum for a vertex weight function w on V. The *MS problem* is the MWS problem where all vertices have the same weight. Let $\alpha_w(G)$ ($\alpha(G)$) denote the maximum weight (maximum cardinality) of a stable vertex set in G.

The M(W)S problem is one of the fundamental algorithmic graph problems which frequently occurs as a subproblem in models in computer science and operations research. It is closely related to the Vertex Cover Problem and to the Maximum Clique Problem in graphs (for an extensive survey on the last one, see [10], which, at the same time, can be seen as a survey on the MWS and the Vertex Cover Problem; however, since 1999, there are many new results on this topic).

The MWS Problem is known to be NP-complete in general and remains NP-complete even on very restricted instances such as $K_{1,4}$-free graphs [48], ($K_{1,4}$,diamond)-free graphs [26], very sparse planar graphs of maximum degree three and graphs not containing cycles below a certain length [53], in particular on triangle-free graphs [55].

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On the other hand, it is known to be solvable in polynomial time on many graph classes by various techniques such as polyhedral optimization, augmenting, struction and other transformations, modular decomposition, bounded clique-width and bounded treewidth, reduction of α-redundant vertices, to mention some basic techniques; for a small selection of papers dealing with particular graph classes and such techniques for M(W)S, see[2-4,8,9,11-19,21-24,27-29,32-40,44,45,48,50-52,58]. Many of these papers deal with subclasses of P_5-free graphs, motivated by the fact that the complexity of the M(W)S problem for P_5-free graphs (and even for (P_5,C_5)-free graphs) is still unknown (for all other 5-vertex graphs H, MS is solvable in polynomial time on (P_5,H)-free graphs). For $2K_2$-free graphs, however, the following is known:

Farber in [30] has shown that a $2K_2$-free graph $G = (V, E)$ contains at most n^2 inclusion-maximal independent sets, $n = |V|$. Thus, the MWS problem on these graphs can be solved in time $\mathcal{O}(n^4)$ since Paull and Unger [54] gave a procedure that generates all maximal independent sets in a graph in $\mathcal{O}(n^2)$ time per generated set (see also [61,43]). This result has been generalized to $l \geq 2$: lK_2-free graphs have at most n^{2l-2} inclusion-maximal independent sets [1,5,31,56], and thus, MWS is solvable on lK_2-free graphs in time $\mathcal{O}(n^{2l})$.

Obviously, the MWS problem on a graph G with vertex weight function w can be reduced to the same problem on antineighborhoods of vertices in the following way:

$$\alpha_w(G) = \max\{w(v) + \alpha_w(G[\overline{N}(v)]) \mid v \in V\}$$

Now, let Π denote a graph property. A graph is *nearly Π* if for each of its vertices, the subgraph induced by the set of its nonneighbors has property Π. (Note that this notion appears in the literature in many variants, e.g., as nearly bipartite graphs [6].)

Thus, whenever MWS is solvable in time T on a class with property Π then it is solvable on nearly Π graphs in time $n \cdot T$. For example, Corneil, Perl and Stewart [27] gave a linear time algorithm for MWS on cographs along the cotree of such a graph. Thus, MWS is solvable in time $\mathcal{O}(nm)$ on nearly cographs. This simple fact, for example, immediately implies Theorem 1 of [32] (which is formulated in [32] for the Maximum Clique Problem and shown there in a more complicated way). For other examples where this approach is helpful, see [14].

A famous divide-and-conquer approach by using clique separators (also called clique cutsets) is described by Tarjan in [60] (see also [62]). For various problems on graphs such as Minimum fill-in, Coloring, Maximum Clique, and the MWS problem, it works well in a bottom-up way along a clique separator tree (which is not uniquely determined but can be constructed in polynomial time for a given graph). The leaves of such a tree, namely the subgraphs not containing clique separators are called *atoms* in [60]. Whenever MWS is solvable in time T on the atoms of a graph G, it is solvable in time $n^2 \cdot T$ on G. However, few examples are known where this approach could be applied for obtaining a polynomial time MWS algorithm on a graph class.

Modular decomposition of graphs is another powerful tool. The decomposition tree is uniquely determined and can be found in linear time [46]. The prime

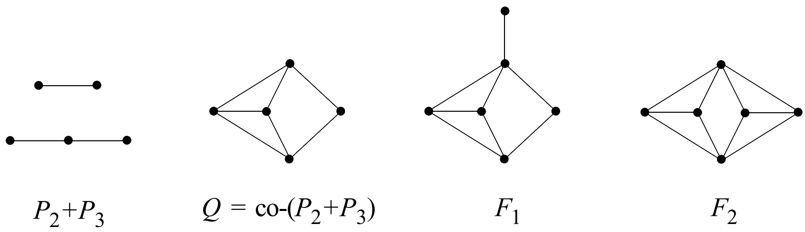


Fig. 1. The $P_2 + P_3$, its complement co-$(P_2 + P_3)$ (called Q), and two Q extensions, called F_1 and F_2

nodes in the tree are the subgraphs having no homogeneous sets (definitions are given later). Again, various problems can be solved efficiently bottom-up along the modular decomposition tree, among them Maximum Clique, and the MWS problem, provided they can be solved efficiently on the prime nodes. In [14], it was shown that a combination of both decompositions is helpful for the MWS problem: If MWS is solvable in time T on prime atoms (i.e., prime subgraphs without clique cutset) of the graph G then it is solvable in time $n^2 \cdot T$ on G.

One of the examples where the clique separator approach works well is given by Alekseev in [3] showing that atoms of (P_5,co-$(P_2 + P_3)$)-free graphs (the $(P_2 + P_3)$ is the graph with five vertices, say a, b, c, d, e and edges ab, cd, de) are $3K_2$-free which implies that the MWS problem is solvable in time $\mathcal{O}(n^8)$ on this graph class (see Figure 1 for the co-$(P_2 + P_3)$).

Our main results in this paper are the following ones:

- (i) Atoms of (P_5,Q)-free graphs are either nearly $(P_5, \overline{P_5}, C_5)$-free or specific (i.e., a simple type of graphs defined later for which the MWS problem can be solved in the obvious way). This leads to an $\mathcal{O}(n^4 m)$ time algorithm for MWS on graphs whose atoms are (P_5,Q)-free which improves and extends Alekseev's result on these graphs [3] (and also the corresponding result of [35] on (P_5, C_5, Q)-free graphs).
- (ii) Prime atoms of (P_5,F_1)-free graphs are $3K_2$-free (see Figure 1 for the F_1). By [14], this implies polynomial time for MWS on (P_5,F_1)-free graphs which extends corresponding polynomial time results on (P_5,Q)-free graphs, on (P_5,co-chair)-free graphs [24], and on (P_5,P)-free graphs [14,22,44] (note, however, that the time bound for (P_5,F_1)-free graphs is much worse than on the last two subclasses mentioned here).
- (iii) Atoms of (P_5,F_2)-free graphs are $4K_2$-free (see again Figure 1 for the F_2). This also extends the result on (P_5,Q)-free graphs.
- (iv) Finally, we show that for every fixed k, MS can be solved in polynomial time for (P_5,H_k)-free graphs (see Figure 3 for the H_k which extends F_1 and F_2).

The first three results give new examples for the power of clique separators.

For space limitations, all proofs are omitted but can be found in the full version of this paper.

2 Basic Notions

Throughout this paper, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges and let $|V| = n$, $|E| = m$. Let $V(G) = V$ denote the vertex set of graph G. For a vertex $v \in V$, let $N(v) = \{u \mid uv \in E\}$ denote the (*open*) *neighborhood* of v in G, let $N[v] = \{v\} \cup \{u \mid uv \in E\}$ denote the (*closed*) *neighborhood* of v in G, and for a subset $U \subseteq V$ and a vertex $v \notin U$, let $N_U(v) = \{u \mid u \in U, uv \in E\}$ denote the *neighborhood* of v with respect to U. The *antineighborhood* $\overline{N}(v)$ is the set $V \setminus N[v]$ of vertices different from v which are nonadjacent to v. We also write $x \sim y$ for $xy \in E$ and $x \not\sim y$ for $xy \notin E$.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph G is $\mathcal{F}$*-free* if none of its induced subgraphs is in $\mathcal{F}$.

A vertex set $U \subseteq V$ is *stable* (or *independent*) in G if the vertices in U are pairwise nonadjacent. For a given graph with vertex weights, the Maximum Weight Stable Set (MWS) Problem asks for a stable set of maximum vertex weight.

Let co-$G = \overline{G} = (V, \overline{E})$ denote the *complement graph* of G. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\overline{G}$. Let K_ℓ denote the clique with ℓ vertices, and let ℓK_1 denote the stable set with ℓ vertices. K_3 is called *triangle*. $G[U]$ is *co-connected* if $\overline{G}[U]$ is connected.

Disjoint vertex sets X, Y *form a join*, denoted by $X \textcircled{1} Y$ (*co-join*, denoted by $X \textcircled{0} Y$) if for all pairs $x \in X$, $y \in Y$, $xy \in E$ ($xy \notin E$) holds. We will also say that X has a join to Y, that there is a join between X and Y, or that X and Y are connected by join (and similarly for co-join). Subsequently, we will consider join and co-join also as operations, i.e., the co-join operation for disjoint vertex sets X and Y is the disjoint union of the subgraphs induced by X and Y (without edges between them), and the join operation for X and Y consists of the co-join operation for X and Y followed by adding all edges $xy \in E$, $x \in X$, $y \in Y$.

A vertex $z \in V$ *distinguishes* vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$ or $zx \notin E$ and $zy \in E$. We also say that *a vertex z distinguishes a vertex set* $U \subseteq V$, $z \notin U$, if z has a neighbor and a non-neighbor in U.

Observation 1. *Let $v \in G[V \setminus U]$ distinguish U.*

(*i*) *If $G[U]$ is connected, then there exist two adjacent vertices $x, y \in U$ such that $v \sim x$ and $v \not\sim y$.*
(*ii*) *If $G[U]$ is co-connected, then there exist two nonadjacent vertices $x, y \in U$ such that $v \sim x$ and $v \not\sim y$.*

A vertex set $M \subseteq V$ is a *module* if no vertex from $V \setminus M$ distinguishes two vertices from M, i.e., every vertex $v \in V \setminus M$ has either a join or a co-join to M. A module is *trivial* if it is $\emptyset$, $V(G)$ or a one-elementary vertex set. A nontrivial module is also called a *homogeneous set*.

A graph G is *prime* if it contains only trivial modules.

The notion of module plays a crucial role in the *modular* (or *substitution*) *decomposition* of graphs (and other discrete structures) which is of basic importance for the design of efficient algorithms - see e.g. [49] for modular decomposition of discrete structures and its algorithmic use and [46] for a linear-time algorithm constructing the modular decomposition tree of a given graph.

A *clique separator* or *clique cutset* in a connected graph G is a clique C such that $G[V \setminus C]$ is disconnected. An *atom* of G is a subgraph of G without clique cutset. See [60] for some algorithmic aspects of the clique separator decomposition.

For $k \geq 1$, let P_k denote a chordless path with k vertices and $k-1$ edges. The $\overline{P_5}$ is also called *house*. For $k \geq 3$, let C_k denote a chordless cycle with k vertices and k edges. A *hole* is a C_k with $k \geq 5$, and an *antihole* is $\overline{C}_k$ with $k \geq 5$. An *odd hole* (*odd antihole*, respectively) is a hole (antihole, respectively) with odd number of vertices.

The $2K_2$ is the co-C_4. More generally, the ℓK_2 consists of 2ℓ vertices, say, $x_1, \ldots, x_\ell$, $y_1, \ldots, y_\ell$ and edges $x_1y_1, \ldots, x_\ell y_\ell$.

For an induced subgraph H of G, a vertex not in H is a *k-vertex* of H, if it has exactly k neighbors in H.

A graph is *chordal* if it contains no induced cycle C_k, $k \geq 4$. A graph is *weakly chordal* if it contains no hole and no antihole. See [20] for a detailed discussion of the importance and the many properties of chordal and weakly chordal graphs. Note that chordal graphs are those graphs whose atoms are cliques.

For a linear order $(v_1, \ldots, v_n)$ of the vertex set V, a well-known coloring heuristic assigns integers to the vertices from left to right such that each vertex v_i gets the smallest positive integer assigned to no neighbor v_j, $j < i$, of v_i. Chvátal defined the important notion of a *perfect order* of a graph $G = (V, E)$ as a linear order $(v_1, \ldots, v_n)$ of V such that for each $k \leq n$, the number of colors used by the preceding coloring heuristic equals the chromatic number of $G[\{v_1, \ldots, v_k\}]$.

A graph is *perfectly orderable* if it has a perfect order. See [20] for various characterizations and properties of these graphs. In particular, recognizing perfectly orderable graphs is NP-complete [47]). A graph G is *perfectly ordered* if a perfect order of G is given. Algorithmic consequences for perfectly orderable graphs rely heavily on this assumption.

3 Atoms of (P_5, Q)-Free Graphs Are Nearly $(P_5, \overline{P_5}, C_5)$-Free or Specific

In this section, we improve the following result:

Theorem 1 (Alekseev [3]). *Atoms of* (P_5, Q)*-free graphs are* $3K_2$*-free.*

Since $3K_2$-free graphs have at most n^4 maximal stable sets, the MWS problem is solvable in time $\mathcal{O}(n^8)$ on (P_5,Q)-free graphs by the clique cutset approach of Tarjan and a corresponding enumeration algorithm for all maximal stable sets in a $3K_2$-free graph. Theorem 1, however, does not give much structural insight.

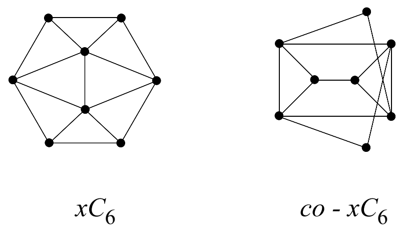


Fig. 2. A two-vertex extension xC_6 of the C_6 and its complement graph, the co-xC_6

Our main result of this section, namely Theorem 2, shows the close connection of (P_5, Q)-free graphs to known classes of perfect graphs and in particular leads to a faster MWS algorithm. Preparing this, we have to define a simple type of graphs which results from a certain extension of the $\overline{C_6}$ by two vertices (which we call $x\overline{C_6}$ or co-xC_6) and the complement of this graph (see Figure 2).

A graph is *specific* if it consists of a co-xC_6 H, a stable set consisting of 2-vertices of H having the same neighbors as one of the degree 2 vertices in H, and a clique U of universal (i.e., adjacent to all other) vertices. Note that the MWS problem for specific graphs can be solved in the obvious way.

Theorem 2. *Atoms of (P_5, Q)-free graphs are either nearly $(P_5, \overline{P_5}, C_5)$-free or specific graphs.*

The proof of Theorem 2 is based on the subsequent Lemmas 1, 2 and 3.

Lemma 1. *Atoms of (P_5, Q)-free graphs are nearly $\overline{P_5}$-free.*

Lemma 2. *Atoms of (P_5, Q)-free graphs containing an induced subgraph $x\overline{C_6}$ are specific graphs.*

Lemma 3. *Atoms of (P_5, Q)-free graphs are either nearly C_5-free or specific graphs.*

In [25], it has been observed that $(P_5, \overline{P_5}, C_5)$-free graphs are perfectly orderable, and a perfect order of such a graph can be constructed in linear time by a degree order of the vertices. Thus, also for $\overline{G}$, a perfect order can be obtained in linear time. In [42], Hoàng gave an $\mathcal{O}(nm)$ time algorithm for the Maximum Weight Clique problem on a perfectly ordered graph (i.e., with given perfect order). This means that the MWS problem on $(P_5, \overline{P_5}, C_5)$-free graphs can be solved in time $\mathcal{O}(nm)$ and consequently, it can be solved on nearly $(P_5, \overline{P_5}, C_5)$-free graphs in time $\mathcal{O}(n^2m)$.

Now, by Theorem 2, MWS is solvable in time $\mathcal{O}(n^2m)$ time on atoms of (P_5,Q)-free graphs. Then the clique separator approach of Tarjan implies:

Corollary 1. *The MWS problem can be solved in time $\mathcal{O}(n^4m)$ on graphs whose atoms are (P_5, Q)-free.*

Note that this class is not restricted to (P_5, Q)-free graphs; it is only required that the atoms are (P_5, Q)-free. Thus, it contains, for example, all chordal graphs. The same remark holds for the other sections.

$(P_5, \overline{P_5}, C_5)$-free graphs are also those graphs which are Meyniel and co-Meyniel (see [20]); Meyniel graphs can be recognized in time $\mathcal{O}(m^2)$ [57]. Thus, nearly $(P_5, \overline{P_5}, C_5)$-free graphs can be recognized in time $\mathcal{O}(n^5)$ (which is even better than $\mathcal{O}(n^4 m)$).

Since $(P_5, \overline{P_5}, C_5)$-free graphs are weakly chordal, another consequence of Theorem 2 is:

Corollary 2. *Atoms of (P_5, Q)-free graphs are either nearly weakly chordal or specific.*

Note that weakly chordal graphs can be recognized in time $\mathcal{O}(m^2)$ [7,41]. Thus, recognizing whether G is nearly weakly chordal can be done in time $\mathcal{O}(nm^2)$. The time bound for MWS on weakly chordal graphs, however, is $\mathcal{O}(n^4)$ [59], and thus, worse than the one for $(P_5, \overline{P_5}, C_5)$-free graphs.

4 Minimal Cutsets in P_5-Free Graphs with ℓK_2

In this section we will collect some useful facts about P_5-free graphs that contain an induced ℓK_2. These facts will be used to prove our main results in Section 5 and represent a more detailed investigation of the background of Alekseev's Theorem 1.

Let $\ell \geq 2$ be an integer, and let G be a P_5-free graph containing an induced $H = \ell K_2$ with $E(H) = \{e_1, e_2, \ldots, e_\ell\}$. Let $S \subseteq V(G) \setminus V(H)$ be an inclusion-minimal vertex set such that, for $i \neq j$, e_i and e_j belong to distinct connected components of $G[V \setminus S]$. S is also called a *minimal cutset* for H. For $1 \leq i \leq \ell$, let H_i be the connected component of $G[V \setminus S]$ containing the edge e_i.

Observation 2.

(i) $\forall v \in S$*:* $N(v) \cap H_i = \emptyset$ *for all* $i \in \{1, 2, \ldots, \ell\}$*, or* $N(v) \cap H_i \neq \emptyset$ *and* $N(v) \cap H_j \neq \emptyset$ *for at least two distinct indices* i, j*.*
(ii) $\forall v \in S$*:* v *distinguishes at most one* H_i*,* $i \in \{1, 2, \ldots, \ell\}$*.*

By Observation 2, S can be partitioned into pairwise disjoint subsets as follows. For $L \subseteq \{1, 2, \ldots, \ell\}$, $|L| \geq 2$, let

$$S_L := \{v \in S \mid (\forall i \in L, N(v) \cap H_i \neq \emptyset) \wedge (\forall j \notin L, N(v) \cap H_j = \emptyset)\},$$

and

$$S_0 := S \setminus \Big(\bigcup_{|L| \geq 2} S_L \Big)$$

as well as

$$R_0 := V \setminus (S \cup H_1 \cup H_2 \cup \cdots \cup H_\ell).$$

Note that $(R_0 \cup S_0) \textcircled{0} (H_1 \cup H_2 \cup \cdots \cup H_\ell)$.

In what follows, L, M, N stand for subsets of $\{1, 2, \ldots, \ell\}$ with at least two elements. Two such subsets are called *incomparable* if each of them is not properly contained in the other. Incomparable sets L, M are *overlapping* if $L \cap M \neq \emptyset$. Note that disjoint sets are mutually incomparable.

Observation 3. *Let L and M be incomparable. Then, for all adjacent vertices $x \in S_L$, $y \in S_M$, $x \textcircled{1} (\bigcup_{i \in L \setminus M} H_i)$ and $y \textcircled{1} (\bigcup_{j \in M \setminus L} H_j)$.*

Observation 4. *Let L and M be overlapping. Then*

(i) $S_L \textcircled{1} S_M$, and
(ii) if $S_L \neq \emptyset$ and $S_M \neq \emptyset$ then $S_L \textcircled{1} (\bigcup_{j \in L \setminus M} H_j)$ and $S_M \textcircled{1} (\bigcup_{i \in M \setminus L} H_i)$.

Observation 5. *Let M be a proper subset of L. Then for all nonadjacent vertices $x \in S_M$, $y \in S_L$,*

(i) $y \textcircled{1} (\bigcup_{i \in L \setminus M} H_i)$, and
(ii) for all $j \in M$, $N(x) \cap H_j \subseteq N(y) \cap H_j$.

Observation 6. *Let $L \cap N = \emptyset$. If some vertex in S_L is nonadjacent to some vertex in S_N, then for all subsets M overlapping with L and with N, $S_M = \emptyset$.*

For each subset $L \subseteq \{1, 2, \ldots, \ell\}$ with at least two elements we partition S_L into pairwise disjoint subsets as follows. Let

$$X_L := \{v \in S_L \mid \forall i \in L, v \textcircled{1} H_i\},$$

and for each $i \in L$,

$$Y_L^i := \{v \in S_L \mid \; v \text{ distinguishes } H_i\}.$$

By Observation 2 (ii),

$$\forall i \in L, Y_L^i \textcircled{1} (\bigcup_{j \in L \setminus \{i\}} H_j) \text{ and } S_L = X_L \cup \bigcup_{i \in L} Y_L^i.$$

Observation 7. *If $|L| \geq 3$ then for all distinct $i, j \in L$, $Y_L^i \textcircled{1} Y_L^j$.*

Observation 8. *If $|L| \geq 3$ and G is F_1-free or F_2-free then $X_L \textcircled{1} (S_L \setminus X_L)$.*

5 (P_5, F_1)-Free and (P_5, F_2)-Free Graphs

Theorem 3. *Prime (P_5, F_1)-free graphs without clique cutset are $3K_2$-free.*

By Theorem 3, prime (P_5, F_1)-free atoms are $3K_2$-free, hence MWS can be solved in time $O(n^5 m)$ on prime (P_5, F_1)-free atoms with n vertices and m edges. Combining with the time bound for MWS via clique separators, we obtain:

Corollary 3. *The MWS problem can be solved in time $O(n^7 m)$ for graphs whose atoms are (P_5, F_1)-free.*

Theorem 4. *(P_5, F_2)-free graphs without clique cutset are $4K_2$-free.*

By Theorem 4, (P_5, F_2)-free atoms are $4K_2$-free, hence MWS can be solved in time $O(n^7 m)$ on (P_5, F_2)-free atoms. Combining again with the clique separator time bound for MWS, we obtain:

Corollary 4. *Maximum Weight Stable Set can be solved in time $O(n^9 m)$ for graphs whose atoms are (P_5, F_2)-free graphs.*

6 Conclusion

In this paper, we give new applications of the clique separator approach, combine it in one case with modular decomposition and extend some known polynomial time results for the Maximum Weight Stable Set problem. In particular, we have shown:

(*i*) Atoms of (P_5,Q)-free graphs are either nearly ($P_5, \overline{P_5}, C_5$)-free or specific which leads to an $\mathcal{O}(n^4m)$ time algorithm for MWS on graphs whose atoms are (P_5,Q)-free improving a result by Alekseev [3].
(*ii*) Prime atoms of (P_5, F_1)-free graphs are $3K_2$-free.
(*iii*) Atoms of (P_5, F_2)-free graphs are $4K_2$-free.

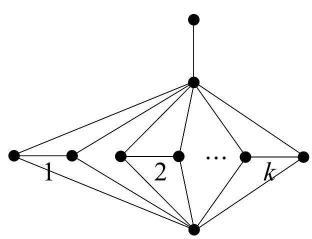
As a consequence, the Maximum Weight Stable Set problem is polynomially solvable for graphs whose atoms are (P_5, F_1)-free ((P_5, F_2)-free, respectively), which tremendously generalizes various polynomially solvable cases known before.

One way in trying to show that the Maximum Weight Stable Set problem can be solved in polynomial time on a large class of P_5-free graphs containing both classes of (P_5, F_1)-free graphs and of (P_5, F_2)-free graphs, is to consider the class of (P_5, H_k)-free graphs, for each fixed integer $k \geq 2$; see Figure 3.

Unfortunately, the technique used in this paper cannot be directly applied for (P_5, H_k)-free graphs. Namely, for each fixed $\ell \geq 3$, there exist *prime* (P_5, H_2)-free graphs that contain an induced ℓK_2 but no clique cutset. However, the unweighted case is easy:

Theorem 5. *For each fixed positive integer k, the Maximum Stable Set problem can be solved in polynomial time for (P_5, H_k)-free graphs.*

Open Problem. Let H_k^- denote the subgraph of H_k without the degree 1 vertex. Is the Maximum *Weight* Stable Set problem solvable in polynomial time for (P_5, H_k^-)-free graphs ($k \geq 3$ fixed)? If yes, the proof of Theorem 5 shows that it is also polynomially solvable for (P_5, H_k)-free graphs, for each fixed positive integer k.



H_k

Fig. 3. The graph H_k

More generally, the following question is of interest: Suppose that MS is polynomially solvable for a certain graph class. Is MWS solvable in polynomial time on the same graph class, too?

References

1. V.E. Alekseev, On the number of maximal stable sets in graphs from hereditary classes, Combinatorial-algebraic Methods in Discrete Optimization, University of Nizhny Novgorod (1991) 5-8 (in Russian)
2. V.E. Alekseev, A polynomial algorithm for finding largest independent sets in fork-free graphs, *Discrete Anal. Oper. Res.* Ser. 1, 6 (1999) 3-19 (in Russian) (see also *Discrete Applied Math.* 135 (2004) 3-16 for the English version)
3. V.E. Alekseev, On easy and hard hereditary classes of graphs with respect to the independent set problem, *Discrete Applied Math.* 132 (2004) 17-26
4. C. Arbib, R. Mosca, On (P_5,diamond)-free graphs, *Discrete Math.* 250 (2002) 1-22
5. E. Balas, Ch.S. Yu, On graphs with polynomially solvable maximum-weight clique problem, *Networks* 19 (1989) 247-253
6. J. Bang-Jensen, J. Huang, G. MacGillivray, A. Yeo, Domination in convex bipartite and convex-round graphs, manuscript 2002
7. A. Berry, J.-P. Bordat, P. Heggernes, Recognizing weakly triangulated graphs by edge separability, *Nordic Journal of Computing* 7 (2000) 164-177
8. P. Bertolazzi, C. De Simone, A. Galluccio, A nice class for the vertex packing problem, *Discrete Applied Math.* 76 (1997) 3-19
9. H. Bodlaender, A. Brandstädt, D. Kratsch, M. Rao, J. Spinrad, Linear time algorithms for some NP-complete problems on (P_5,gem)-free graphs, FCT'2003, LNCS 2751, 61-72 (2003)
10. I.M. Bomze, M. Budinich, P.M. Pardalos, M. Pelillo, The maximum clique problem. In: D.-Z. Du, P.M. Pardalos, eds., *Handbook of Combinatorial Optimization, Supplement Volume A*, Kluwer Academic Press, Boston, MA, USA (1999)
11. A. Brandstädt, (P_5,diamond)-Free Graphs Revisited: Structure and Linear Time Optimization, *Discrete Applied Math.* 138 (2004) 13-27
12. A. Brandstädt, F.F. Dragan, On the linear and circular structure of (claw,net)-free graphs, *Discrete Applied Math.* 129 (2003) 285-303
13. A. Brandstädt, P.L. Hammer, On the stability number of claw-free P_5-free and more general graphs, *Discrete Applied Math.* 95 (1999) 163-167
14. A. Brandstädt, C.T. Hoàng, On clique separators, nearly chordal graphs, and the Maximum Weight Stable Set Problem, manuscript (2004); accepted for IPCO 2005, Berlin
15. A. Brandstädt, C.T. Hoàng, V.B. Le, Stability Number of Bull- and Chair-Free Graphs Revisited, *Discrete Applied Math.* 131 (2003) 39-50
16. A. Brandstädt, D. Kratsch, On the structure of (P_5,gem)-free graphs; *Discrete Applied Math.* 145 (2005) 155-166
17. A. Brandstädt, V.B. Le, H.N. de Ridder, Efficient Robust Algorithms for the Maximum Weight Stable Set Problem in Chair-Free Graph Classes, *Information Processing Letters* 89 (2004) 165-173
18. A. Brandstädt, H.-O. Le, V.B. Le, On α-redundant vertices in P_5-free graphs, *Information Processing Letters* 82 (2002) 119-122
19. A. Brandstädt, H.-O. Le, R. Mosca, Chordal co-gem-free graphs and (P_5,gem)-free graphs have bounded clique-width, *Discrete Applied Math.* 145 (2005) 232-241

20. A. Brandstädt, V.B. Le, and J.P. Spinrad, Graph Classes: A Survey, *SIAM Monographs on Discrete Math. Appl., Vol.* 3, SIAM, Philadelphia (1999)
21. A. Brandstädt, H.-O. Le, J.-M. Vanherpe, Structure and Stability Number of (Chair,Co-P,Gem)-Free Graphs Revisited, *Information Processing Letters* 86 (2003) 161-167
22. A. Brandstädt and V.V. Lozin, A note on α-redundant vertices in graphs, *Discrete Applied Math.* 108 (2001) 301-308
23. A. Brandstädt, S. Mahfud, Maximum Weight Stable Set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time, *Information Processing Letters* 84 (2002) 251-259
24. A. Brandstädt, R. Mosca, On the structure and stability number of P_5- and co-chair-free graphs, *Discrete Applied Math.* 132 (2004) 47-65
25. V. Chvátal, C.T. Hoàng, N.V.R. Mahadev, D. de Werra, Four classes of perfectly orderable graphs, *J. Graph Theory* 11 (1987) 481-495
26. D.G. Corneil, The complexity of generalized clique packing, *Discrete Applied Math.* 12 (1985) 233-240
27. D.G. Corneil, Y. Perl, and L.K. Stewart, Cographs: recognition, applications, and algorithms, *Congressus Numer.* 43 (1984) 249-258
28. C. De Simone, On the vertex packing problem, *Graphs and Combinatorics* 9 (1993) 19-30
29. C. De Simone, A. Sassano, Stability number of bull- and chair-free graphs, *Discrete Applied Math.* 41 (1993) 121-129
30. M. Farber, On diameters and radii of bridged graphs, *Discrete Math.* 73 (1989) 249-260
31. M. Farber, M. Hujter, Zs. Tuza, An upper bound on the number of cliques in a graph, *Networks* 23 (1993) 207-210
32. M.U. Gerber, A. Hertz, A transformation which preserves the clique number, *J. Combin. Th.* (B) 83 (2001) 320-330
33. M.U. Gerber, A. Hertz, D. Schindl, P_5-free augmenting graphs and the maximum stable set problem, *Discrete Applied Math.* 132 (2004) 109-119
34. M.U. Gerber, V.V. Lozin, Robust algorithms for the stable set problem, *Graphs and Combinatorics* 19 (2003) 347-356
35. M.U. Gerber, V.V. Lozin, On the stable set problem in special P_5-free graphs, *Discrete Applied Math.* 125 (2003) 215-224
36. V. Giakoumakis, I. Rusu, Weighted parameters in $(P_5, \overline{P_5})$-free graphs, *Discrete Applied Math.* 80 (1997) 255-261
37. M.C. Golumbic, P.L. Hammer, Stability in circular-arc graphs, *J. Algorithms* 9 (1988) 314-320
38. M. Grötschel, L. Lovász, and A. Schrijver, The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* 1 (1981) 169-197
39. P.L. Hammer, N.V.R. Mahadev, and D. de Werra, Stability in CAN-free graphs, *J. Combin. Th.* (B) 38 (1985) 23-30
40. P.L. Hammer, N.V.R. Mahadev, and D. de Werra, The struction of a graph: Application to CN-free graphs, *Combinatorica* 5 (1985) 141-147
41. R. Hayward, J.P. Spinrad, R. Sritharan, Weakly chordal graph algorithms via handles, *Proceedings* of 11th SODA'2000, 42-49 (2000)
42. C.T. Hoàng, Efficient algorithms for minimum weighted colouring of some classes of perfect graphs, *Discrete Applied Math.* 55 (1994) 133-143
43. D.S. Johnson, M. Yannakakis, C.H. Papadimitriou, On generating all maximal independent sets, *Information Processing Letters* 27 (1988) 119-123

44. V.V. Lozin, Stability in P_5- and banner-free graphs, *European J. Oper. Res.* 125 (2000) 292-297
45. V.V. Lozin, R. Mosca, Independent sets in extensions of $2K_2$-free graphs, *Discrete Applied Math.* 146 (2005) 74-80
46. R.M. McConnell, J. Spinrad, Modular decomposition and transitive orientation, *Discrete Math.* 201 (1999) 189-241
47. M. Middendorf, F. Pfeiffer, On the complexity of recognizing perfectly orderable graphs, *Discrete Math.* 80 (1990) 327-333
48. G.M. Minty, On Maximal Independent Sets of Vertices in Claw-Free Graphs, *J. Combin. Theory* (B) 28 (1980) 284-304
49. R.H. Möhring, F.J. Radermacher, Substitution decomposition for discrete structures and connections with combinatorial optimization, *Annals of Discrete Math.* 19 (1984) 257-356
50. R. Mosca, Polynomial algorithms for the maximum stable set problem on particular classes of P_5-free graphs, *Information Processing Letters* 61 (1997) 137-143
51. R. Mosca, Stable sets in certain P_6-free graphs, *Discrete Applied Math.* 92 (1999) 177-191
52. R. Mosca, Some results on maximum stable sets in certain P_5-free graphs, *Discrete Applied Math.* 132 (2003) 175-184
53. O.J. Murphy, Computing independent sets in graphs with large girth, *Discrete Applied Math.* 35 (1992) 167-170
54. M. Paull, S. Unger, Minimizing the number of states in incompletely specified sequential switching functions, *IRE Transactions on Electronic Computers* 8 (1959) 356-367
55. S. Poljak, A note on stable sets and colorings of graphs, *Commun. Math. Univ. Carolinae* 15 (1974) 307-309
56. E. Prisner, Graphs with few cliques, Graph Theory, Combinatorics and Algorithms, Kalamazoo, MI, 1992, 945-956, Wiley, New York, 1995
57. F. Roussel, I. Rusu, Holes and dominoes in Meyniel graphs, *Internat. J. Foundations of Comput. Sci.* 10 (1999) 127-146
58. N. Sbihi, Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile, *Discrete Applied Math.* 29 (1980) 53-76
59. J.P. Spinrad, R. Sritharan, Algorithms for weakly chordal graphs, *Discrete Applied Math.* 19 (1995) 181-191
60. R.E. Tarjan, Decomposition by clique separators, *Discrete Math.* 55 (1985) 221-232
61. S. Tsukiyama, M. Ide, H. Ariyoshi, I. Shirakawa, A new algorithm for generating all the maximal independent sets, *SIAM J. Computing* 6 (1977) 505-517
62. S.H. Whitesides, A method for solving certain graph recognition and optimization problems, with applications to perfect graphs, in: Annals of Discrete Math. 21 (1984) Berge, C. and V. Chvátal (eds), *Topics on perfect graphs*, North-Holland, Amsterdam, 1984, 281-297