Available online at www.sciencedirect.com
SCIENCE DIRECT®
Information Processing Letters 98 (2006) 133–138
Information Processing Letters
www.elsevier.com/locate/ipl

Structure and linear time recognition of 3-leaf powers

Andreas Brandstädt *, Van Bang Le

Institut für Informatik, Universität Rostock, D-18051 Rostock, Germany

Received 24 September 2005; received in revised form 5 January 2006; accepted 11 January 2006

Available online 3 February 2006

Communicated by A.A. Bertossi

Abstract

A graph G is the *k-leaf power* of a tree T if its vertices are leaves of T such that two vertices are adjacent in G if and only if their distance in T is at most k. Then T is the *k-leaf root* of G. This notion was introduced and studied by Nishimura, Ragde, and Thilikos motivated by the search for underlying phylogenetic trees. Their results imply a $O(n^3)$ time recognition algorithm for 3-leaf powers. Later, Dom, Guo, Hüffner, and Niedermeier characterized 3-leaf powers as the (bull, dart, gem)-free chordal graphs. We show that a connected graph is a 3-leaf power if and only if it results from substituting cliques into the vertices of a tree. This characterization is much simpler than the previous characterizations via critical cliques and forbidden induced subgraphs and also leads to linear time recognition of these graphs.

Keywords: 3-Leaf powers; 3-Leaf roots; Leaf powers; Leaf roots; Linear time recognition algorithm; Graph algorithms

1. Introduction

As Nishimura et al. mention in [8], "a fundamental problem in computational biology is the reconstruction of the *phylogeny*, or evolutionary history, of a set of species or genes, typically represented as a *phylogenetic tree* ...". Motivated by this background, in [8], the crucial notion of *k-leaf power* and *k-leaf root* is defined as follows:

Let $G = (V, E)$ be a finite undirected graph. G is a *k-leaf power* if there is a tree T with V as leaves such that for all $x, y \in V$, $xy \in E$ if and only if their distance in T is at most k: $d_T(x, y) \leqslant k$. T is then called a *k-leaf root* of G.

Obviously, a graph is a 2-leaf power if and only if it is the disjoint union of cliques, i.e., it contains no induced P_3.

Nishimura et al. [8] give (very complicated) $O(n^3)$ time recognition algorithms for recognizing 3-leaf powers and 4-leaf powers, respectively, and constructing 3-leaf roots (4-leaf roots, respectively), if existent. Their algorithm relies on the concept of the (directed) clique graph of a graph. For $k \geqslant 5$, no characterization of k-leaf powers and no efficient recognition is known.

In [4], Dom et al. give a forbidden subgraph characterization of 3-leaf powers which, however, does not lead to a faster recognition of 3-leaf powers. A basic tool in [4] is the concept of critical cliques of a graph introduced in [6].

We will avoid the construction of (directed or critical) clique graphs and give a new characterization of 3-leaf powers which, among various others, leads to linear time recognition of these graphs.

* Corresponding author.
E-mail addresses: ab@informatik.uni-rostock.de (A. Brandstädt), le@informatik.uni-rostock.de (V.B. Le).

0020-0190/$ – see front matter
doi:10.1016/j.ipl.2006.01.004

2. Basic notions

Throughout this note, let $G = (V, E)$ be a finite undirected graph without self-loops and multiple edges with vertex set V and edge set E, and let $|V| = n$, $|E| = m$. For a vertex $v \in V$, let $N_G(v) = N(v) = \{u \mid uv \in E\}$ denote the (*open*) *neighborhood* of v in G, and let $N_G[v] = N[v] = \{v\} \cup \{u \mid uv \in E\}$ denote the *closed neighborhood* of v in G. A *clique* is a set of vertices which are mutually adjacent. A *stable set* is a set of vertices which are mutually nonadjacent.

Two vertices $x, y \in V$ are *true twins* if $N[x] = N[y]$, i.e., x and y have the same neighbors and are adjacent to each other. Two vertices are *false twins* if they have the same neighbors and are nonadjacent to each other. The *true twin operation* (*false twin operation*, respectively) adds a new vertex y to graph G which is a true twin (false twin, respectively) to an already existing vertex x in G. The *pendant vertex operation* adds a new vertex y being adjacent to only one vertex x in G.

A vertex subset $U \subseteq V$ is a *module* in G if for all $v \in V \setminus U$, either v is adjacent to all vertices of U or v is adjacent to none of them. A *clique module* in G is a module which induces a clique in G. Obviously, a set of vertices that are pairwise true twins are a clique module.

Let $d_G(x, y)$ (or $d(x, y)$ for short if G is understood) be the length, i.e., number of edges, of a shortest path in G between x and y. Let $N_G^k(x) = \{y \mid d_G(x, y) = k\}$ and let $G^k = (V, E^k)$ with $xy \in E^k$ if and only if $d_G(x, y) \leqslant k$ denote the *kth power of G*.

For $U \subseteq V$, let $G[U]$ denote the subgraph of G induced by U. Throughout this paper, all subgraphs are understood to be induced subgraphs. Let $\mathcal{F}$ denote a set of graphs. A graph G is *$\mathcal{F}$-free* if none of its induced subgraphs is in $\mathcal{F}$.

For $k \geqslant 1$, let P_k denote a chordless path with k vertices and $k - 1$ edges, and for $k \geqslant 3$, let C_k denote a chordless cycle with k vertices and k edges.

A graph is *chordal* if it contains no induced C_k, $k \geqslant 4$. A vertex is *simplicial* in G if its neighborhood $N[v]$ is a clique. A vertex ordering $(v_1, \dots, v_n)$ is a *perfect elimination ordering* (*p.e.o.*) of G if for every $i \in \{1, \dots, n\}$, v_i is simplicial in the subgraph $G_i = G[\{v_i, \dots, v_n\}]$. It is well known that a graph is chordal if and only if it has a p.e.o. [11].

A graph is *strongly chordal* if it is chordal and "sun-free"—see [2] for the definition of a sun and for various characterizations of strongly chordal graphs. A vertex is *simple* in G [5] if the closed neighborhoods of all vertices $x, y \in N[v]$ are pairwise comparable with respect to set inclusion. Every simple vertex is simplicial [5]. A vertex ordering $(v_1, \dots, v_n)$ is a *simple elimination ordering* (*s.e.o.*) of G if for every $i \in \{1, \dots, n\}$, v_i is simple in the subgraph $G_i = G[\{v_i, \dots, v_n\}]$. Farber [5] has shown that a graph is strongly chordal if and only if it has a s.e.o.

3. Some basic facts on *k*-leaf powers

The following facts on k-leaf powers are well known.

Proposition 1.

(i) *Every induced subgraph of a k-leaf power is a k-leaf power.*
(ii) *A graph is a k-leaf power if and only if each of its connected components is a k-leaf power.*

Proof. (i) Let T be a k-leaf root of a graph G, and let H be an induced subgraph of G. Then, by definition, the tree T' obtained from T by deleting the leaves which correspond to vertices in $G - H$ is a k-leaf root of H.

(ii) The only-if part follows from (i). For the if-part, assume that each connected component G_i of G has a k-leaf root T_i. Take a new vertex v and connect the trees T_i and v by a path of length k; the resulting tree is clearly a k-leaf root of G. □

In [3,7,10], it is shown that the class of strongly chordal graphs is closed under powers:

Proposition 2. ([3,7,10]) *If G is strongly chordal then for every $k \geqslant 1$, G^k is strongly chordal.*

Let T be a k-leaf root of a graph G. Then, by definition, G is isomorphic to the subgraph of T^k induced by the leaves of T. Since trees are strongly chordal and induced subgraphs of strongly chordal graphs are strongly chordal, Proposition 2 implies:

Proposition 3. *For every $k \geqslant 1$, k-leaf powers are strongly chordal.*

This strengthens the fact that k-leaf powers are chordal which is observed in previous papers dealing with k-leaf powers. The following facts are likely to be known.

Proposition 4. *Every k-leaf power is a $(k + 2)$-leaf power.*

Proof. Let T be a k-leaf root of G, and let T' be the tree obtained from T by subdividing each pendant edge with a new vertex. Thus, the leaves of T' are exactly those of T. Clearly, for all $x, y \in V(G)$, $xy \in E(G)$ if and only if $k \geqslant d_T(x, y) = d_{T'}(x, y) - 2$, hence T' is a $(k+2)$-leaf of G. □

We do not know, in general, if any k-leaf power is also a $(k+1)$-leaf power. For 3-leaf powers, however, we have:

Proposition 5. *Every* 3*-leaf power is a* k*-leaf power for all* $k \geqslant 3$.

Proof. Let T be a 3-leaf root of a graph G, and let T' be the tree obtained from T by subdividing each non-pendant edge with exactly $k-3$ new vertices. Thus, the leaves of T' are exactly those of T. Clearly, for all $x, y \in V(G)$, $xy \in E(G)$ if and only if $d_T(x, y) = d_{T'}(x, y) = 2$, or $d_T(x, y) = 3$ and $d_{T'}(x, y) = k$. Hence T' is a k-leaf of G. □

Substituting a vertex v in a graph G by a graph H results in the graph obtained from $(G - v) \cup H$ by adding all edges between vertices in $N_G(v)$ and vertices in H.

Proposition 6. *For every graph* G*, and for every* $k \geqslant 2$*,* G *is a* k*-leaf power if and only if every graph obtained from* G *by substituting the vertices by cliques is a* k*-leaf power.*

Proof. If there is a k-leaf root T for graph $G = (V, E)$, i.e., T is a tree with leaf set V such that for all $x, y \in V$, $xy \in E$ if and only if $d_T(x, y) \leqslant k$ and G is the result of substituting a vertex $u \in V$ by a clique Q then attach all Q vertices at the same T parent as u; the resulting tree T' is a k-leaf root for G'. The converse direction obviously holds. □

4. Characterizations of 3-leaf powers

In [4], 3-leaf powers were characterized as the chordal (bull, dart, gem)-free graphs (see Fig. 1 for the bull, dart and gem).

In this section, we obtain new characterizations of 3-leaf powers which lead to linear time recognition of these graphs. For this purpose, we collect some further basic facts.

Observation 7. *Every tree is a* 3*-leaf power, and a* 3*-leaf root can be determined in linear time.*

Proof. For a tree $T = (V, E)$, let T' be the tree obtained from T by adding a copy v' for each $v \in V$ (i.e., the new vertices v' are the leaves of T') and adding the new edges vv' to E. Clearly, $xy \in E$ if and only if $d_{T'}(x', y') \leqslant 3$. □

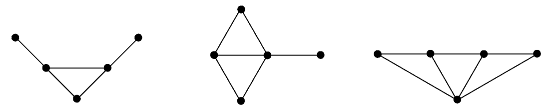

Fig. 1. The bull, dart and gem have no 3-leaf root.

By Proposition 1(ii), we obtain:

Corollary 8. *Every forest is a* 3*-leaf power, and a* 3*-leaf root of it can be determined in linear time.*

Thus, by Proposition 6, graphs obtained from a forest by substituting the vertices by cliques are 3-leaf powers. We will see that these graphs are exactly the 3-leaf powers. By Proposition 1(ii), we can assume that the graph is connected.

Theorem 9. *A connected graph* G *is a* 3*-leaf power if and only if* G *is the result of substituting cliques into the vertices of a suitable tree.*

Proof. First assume that the connected graph $G = (V, E)$ is a 3-leaf power, and let T be a 3-leaf root for G with leaf set V. Let $V = V_1 \cup \cdots \cup V_k$ be a partition of V with respect to common parent nodes in T, i.e., V_i is the (nonempty) set of T-leaves in V having the same parent node t_i in T.

Claim 10. *If for all* $i \in \{1, \ldots, k\}$, $|V_i| = 1$ *then* G *is a tree.*

Proof. Assume not. Since G, as a 3-leaf power, is chordal, G contains a C_3, say with vertices a, b, c. Let x' be the parent node in T for $x \in \{a, b, c\}$. Then a', b', c' are pairwise different, and since $ab \in E$, $bc \in E$ and $ac \in E$, i.e., their distance in T is at most 3, $a'b'$, $b'c'$ and $c'a'$ must be edges in T, i.e., the tree T contains a C_3—contradiction. This shows Claim 10. □

Obviously, the sets V_i, $i \in \{1, \ldots, k\}$, are clique modules in G. If G does not fulfill the assumption that for all $i \in \{1, \ldots, k\}$, $|V_i| = 1$, then let G^* be the induced subgraph of G by taking a representative vertex from each of the nonempty sets V_i, $i \in \{1, \ldots, k\}$. For G^*, Claim 10 applies, i.e., G^* is a tree. Now it is easy to see that G is the result of substituting the cliques V_i into the corresponding representative vertices in G^*.

Conversely, assume that G is the result of substituting the vertices of a suitable tree G^* by some cliques.

Then by Observation 7, G^* is a 3-leaf power, and by Proposition 6, G is also a 3-leaf power which shows the theorem. □

The *distance-hereditary graphs* were characterized in [1] as the graphs obtained from a single vertex by repeatedly applying the pendant vertex, true twin and false twin operations. The graphs which are distance hereditary and chordal are exactly the gem-free chordal graphs (see e.g. [2]). Thus, 3-leaf powers are a subclass of distance-hereditary graphs. More exactly, one can show the following:

Corollary 11. *A connected graph G is a 3-leaf power if and only if G is the result of a sequence of pendant vertex operations, starting with a single vertex, followed by a sequence of true twin operations.*

Proof. First assume that G is a 3-leaf power. Then, by Theorem 9, G is the result of substituting cliques into the vertices of a tree T. Let σ_1 be a sequence of pendant vertex operations which generate T. Now the substitution of cliques into T which produces G can be done by a sequence σ_2 of true twin operations. Thus, G results from a sequence of pendant vertex operations followed by a sequence of true twin operations.

Conversely assume that G results from a sequence σ_1 of pendant vertex operations, followed by a sequence σ_2 of true twin operations. Then σ_1 generates a tree T, and σ_2 produces a set of cliques which are substituted into the vertices of T. Again by Theorem 9, G is a 3-leaf power. □

Note that if pendant vertex and true twin operations are mixed then also graphs can be generated which are no 3-leaf powers; the bull and the dart are such examples. In particular, the class of 3-leaf powers is not closed under the pendant vertex operation.

Corollary 11 can also be formulated in the following way:

Corollary 12. *A graph G is a 3-leaf power if and only if every induced subgraph of G is a forest or has true twins.*

The following Theorem 13 which characterizes 3-leaf powers in terms of forbidden subgraphs was already given in [4] using the notion of critical cliques. We will give a simpler proof for it.

Theorem 13. *A graph G is a 3-leaf power if and only if G is bull-, dart-, and gem-free chordal.*

Proof. If G is a 3-leaf power then by Theorem 9, its connected components result from a tree by substituting cliques into its vertices. The reader can easily verify that such graphs are bull-, dart-, and gem-free chordal.

Conversely assume that G is bull-, dart-, and gem-free chordal. We use Corollary 12 in order to show that G is a 3-leaf power. It suffices to prove that G itself is a forest or has true twins since all induced subgraphs of G are bull-, dart-, and gem-free chordal. Suppose that G is not a forest. Then, as G is chordal, G has a maximal clique Q with at least three vertices. If Q has no neighbor in $G - Q$ then clearly, every two vertices in Q form true twins. Now, let $v \in G - Q$ be adjacent to a vertex $q_1 \in Q$. As Q is a maximal clique, there is a vertex $q_2 \in Q$ such that v is nonadjacent to q_2. Consider a vertex $q_3 \in Q \setminus \{q_1, q_2\}$.

Now, if v is nonadjacent to q_3 then q_2, q_3 form true twins since otherwise, if a vertex u distinguishes q_2 and q_3, say u is adjacent to q_2 and nonadjacent to q_3 then u, v, q_1, q_2, q_3 would induce a bull or dart or gem, or u, v, q_1, q_2 would induce a C_4 (depending on the possible edges uq_1, uv).

If v is adjacent to q_3 then q_1, q_3 form true twins since otherwise, if a vertex u distinguishes q_1 and q_3, say u is adjacent to q_1 and nonadjacent to q_3 then u, v, q_1, q_2, q_3 would induce a dart or gem, or u, v, q_2, q_3 would induce a C_4 (depending on the possible edges uq_2, uv). □

It might be an interesting question whether 3-leaf powers can be characterized in terms of special s.e.o. The following observations illustrate this. For a simple vertex $v \in V$, we classify the vertices in $N[v]$ with respect to their closed neighborhood: Two vertices $x, y \in N[v]$ are equivalent ($x \sim y$) if $N[x] = N[y]$. Note that if a graph G is a 3-leaf power then G has a simple elimination ordering $(v_1, \ldots, v_n)$ such that for all $i \in \{1, \ldots, n\}$, the vertices in $N[v_i]$ have only two equivalence classes with respect to $\sim$ since, by Theorem 13, G is (bull, dart, gem)-free chordal: Let $(v_1, \ldots, v_n)$ be a p.e.o. of G. Since induced subgraphs of G are again (bull, dart, gem)-free chordal, it suffices to discuss v_1. Every simplicial vertex in G is also simple in G since G is bull-free and chordal. Now assume that there are vertices $x, y \in N[v_1]$ with $N[v_1] \subset N[x] \subset N[y]$. Let $z \in N[x] \cap N[y] \setminus N[v_1]$ and $u \in N[y] \setminus (N[v_1] \cup N[x])$. Then v_1, x, y, z, u induce a dart or gem—a contradiction.

The converse direction, however, does not work as the example of the bull shows. We leave it as an open problem to characterize 3-leaf powers in terms of special s.e.o.

The following Theorem 14 summarizes the various characterizations of 3-leaf powers described above, and adds one more condition which leads to linear time recognition. Recall from Section 2 that for $i \geqslant 0$, $N_G^i(v) = \{u \mid d_G(v,u) = i\}$. Thus, in particular, $N_G^0(v) = \{v\}$, and $N_G^1(v) = N_G(v)$. By Proposition 1(ii), we can assume that G is connected.

Theorem 14 *(Structure theorem for 3-leaf powers). For every connected graph G, the following conditions are equivalent*:

(i) *G is a 3-leaf power.*
(ii) *G is bull-, dart-, and gem-free chordal.*
(iii) *G is the result of a sequence of pendant vertex operations, starting with a single vertex, followed by a sequence of true twin operations.*
(iv) *Every induced subgraph of G is a forest or has true twins.*
(v) *G is obtained from a tree T by substituting the vertices of T by cliques.*
(vi) *G is chordal, and for every simplicial vertex v of G the following conditions hold for any $i \geqslant 1$ where N_i stands for $N_G^i(v)$:*
   (a) *Each connected component of $G[N_i]$ is a clique*;
   (b) *For every $x \in N_i$, $N(x) \cap N_{i-1}$ is a clique module in G. Moreover, if $i \geqslant 3$ then $N(x) \cap N_{i-1}$ is a connected component of $G[N_{i-1}]$*;
   (c) *For all $x, y \in N_i$, if x, y belong to the same connected component in $G[N_i]$ or $i = 2$, then $N(x) \cap N_{i-1} = N(y) \cap N_{i-1}$*;
   (d) *For all $x, y \in N_i$, if x, y belong to different connected components in $G[N_i]$ then $N(x) \cap N_{i-1} = N(y) \cap N_{i-1}$ whenever x and y have a common neighbor in N_{i-1}.*

Proof. The equivalence (i) ⇔ (ii) is given in Theorem 13.

The equivalence (i) ⇔ (iii) is given in Corollary 11.

The equivalence (i) ⇔ (iv) is given in Corollary 12.

The equivalence (i) ⇔ (v) is given in Theorem 9.

(i) ⇒ (vi) By the equivalence of (i) and (v), we can assume that G fulfills (v). Thus, let G be obtained from a tree T by substituting vertices $a \in V(T)$ by cliques C_a. Then G is clearly chordal (indeed, graphs obtained from a chordal graph by substituting vertices by cliques are chordal). Also, it is clear that $v \in V(G)$ is a simplicial vertex in G if and only if $v \in C_a$ for a leaf a in T. Let $v \in V(G)$ be a simplicial vertex in G and let a be the leaf of T such that $v \in C_a$, and consider T as rooted at vertex a. From the assumption on G we have the following facts:

- For all $b \in V(T)$, C_b is a (clique) module in G;
- $N_G^1(v) = (C_a - v) \cup C_{a'}$ induces a clique in G where a' is the father of a in T;
- For all $i \geqslant 2$, $N_G^i(v)$ consists of the disjoint cliques C_b, $b \in N_T^i(a)$;
- For all $i \geqslant 2$, for all $b \in N_T^i(a)$, for all $x, y \in C_b$: $N_G(x) \cap N_G^{i-1}(v) = C_{b'} = N_G(y) \cap N_G^{i-1}(v)$ where $b' \in N_T^{i-1}(a)$ is the father of b in T;
- For all $i \geqslant 2$, for all distinct $b_1, b_2 \in N_T^i(a)$, for all $x \in C_{b_1}$, $y \in C_{b_2}$: If b_1 and b_2 have the same father $d \in N_T^{i-1}(a)$, then $N_G(x) \cap N_G^{i-1}(v) = C_d = N_G(y) \cap N_G^{i-1}(v)$.

From these facts, the conditions (a)–(d) follow directly.

(vi) ⇒ (i) Let G satisfy (vi) with an arbitrary simplicial vertex v. Let p be such that $N_p \neq \emptyset$ and $V(G) = N_0 \cup N_1 \cup \cdots \cup N_p$. A 3-leaf root of G can be constructed as follows. Let $A := N(N_2) \cap N_1$ and $B := \{v\} \cup (N_1 \setminus A)$. Let H be the graph obtained from G by contracting A, B to a single vertex a, respectively, b, and each connected component $C_j^{(i)}$ in $G[N_i]$ to a single vertex $c_j^{(i)}$, $2 \leqslant i \leqslant p$. By (a)–(d), H is a tree. Let T be the tree obtained from H by attaching at each vertex $c_j^{(i)}$ the set $C_j^{(i)}$ of leaves, and at a, b the sets A and B, respectively, of leaves. By (a)–(d) again, it is easily seen that T is a 3-leaf root of G. □

5. Linear time recognition of 3-leaf powers

Theorem 14, (i) ⇔ (vi) leads to the following linear time algorithm that decides if a given graph is a 3-leaf power, and, if so, outputs a 3-leaf root. Note that by Proposition 1(ii), we may consider connected graphs only.

Condition (v) of Theorem 14 reminds of the characterization of distance hereditary graphs given in [1] as the result of pendant vertex, true twin and false twin operations applied to a single vertex.

Algorithm. `3-Leaf-Power`
Input: A connected graph $G = (V, E)$ with $|V| = n$ and $|E| = m$.
Output: A 3-leaf root T of G if one exists; otherwise 'NO'.

if G is chordal **then**
begin
  determine a simplicial vertex v;
  compute the sets $N_G^1(v), \ldots, N_G^p(v)$;

```
 (5)      compute the connected component
              of G[N^i_G(v)], 2 ⩽ i ⩽ p;
 (6)      if for some 1 ⩽ i ⩽ p, one of the conditions (a)–(d)
              in Theorem 14(vi) fails
 (7)      then return 'NO' and STOP;
 (8)      return T according to the proof
              of (vi) ⇒ (i) in Theorem 14
 (9)   end
(10)   else return 'NO'.
```

Theorem 15. *Algorithm* `3-Leaf-Power` *is correct and takes linear time.*

Proof. The correctness of the algorithm follows from Theorem 14. The time bound is $O(n+m)$ since recognition of chordal graphs can be done in linear time [11], and a simplicial vertex can be determined in linear time, step (4) can be done in linear time by using Breadth-First Search, and then, by the usual techniques, (a)–(d) can be checked in linear time. If all of them are fulfilled, the contraction of the corresponding sets as described in the proof of (vi) $\Rightarrow$ (i) in Theorem 14 can be done in linear time. □

6. Uniqueness of 3-leaf roots

Finally, we address the natural question how many 3-leaf roots a 3-leaf power may have. Obviously, cliques and disconnected 3-leaf powers have many (non-isomorphic) 3-leaf roots. We close our discussion on 3-leaf powers by the following uniqueness properties.

Lemma 16. *Every tree with at least three vertices has a unique* 3*-leaf root.*

Proof. Let B be a tree with at least 3 vertices, and let T be an arbitrary 3-leaf root of B. Recall that $V(B)$ is exactly the set of leaves of T. We first prove:

$$\text{Every vertex } t \in T \setminus V(B) \text{ is a father of exactly one leaf } b \in V(B). \tag{1}$$

For, if t is not the father of any leaf $b \in V(B)$, B would be disconnected. If t is the father of two leaves $b_1 \neq b_2$ in $V(B)$, b_1, b_2 would form true twins in B, and B would have a triangle containing b_1, b_2.

By (1) and by definition of T, the mapping $b \mapsto t :=$ father of b in T is a bijection such that $bb' \in E(B)$ if and only if $tt' \in E(T \setminus V(B))$. That is, $T \setminus V(B)$ is isomorphic to B. This and (1) show that, given B, the 3-leaf root T of B is unique. This shows Lemma 16. □

Theorem 17. *Every connected* 3*-leaf power different from a clique has a unique* 3*-leaf root.*

Proof. Let G be a connected 3-leaf power that is not a clique, and let T be an arbitrary 3-leaf root of G. We will show by induction that T is unique. If G is a tree, T is unique by Lemma 16. Otherwise, by Corollary 11, G has true twins x, y. As G has more than two vertices, x and y must have the same father in T. As G is connected and not a clique, $G - y$ is connected and not a clique, too. By induction, $T - y$ is the unique 3-leaf root of $G - y$. Since y is uniquely determined by the father of x, T is therefore the unique 3-leaf root of T. □

Note that for every $k \geqslant 4$, k-leaf roots of a connected non-clique k-leaf power are not unique in general.

7. Conclusion

Another way of giving a linear time recognition of 3-leaf powers is modular decomposition; however, we prefer our approach expressed in Algorithm `3-Leaf-Power` since it is conceptually much simpler and more practical.

Open problem. Characterization and polynomial time recognition of k-leaf powers for $k \geqslant 5$.

After finishing this note we learnt that Rautenbach [9] independently found the result described in Theorem 9.

References

[1] H.J. Bandelt, H.M. Mulder, Distance hereditary graphs, J. Combin. Theory (B) 41 (1986) 182–208.

[2] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: A Survey, SIAM Monographs on Discrete Math. Appl., vol. 3, SIAM, Philadelphia, 1999.

[3] E. Dahlhaus, P. Duchet, On strongly chordal graphs, Ars Combin. B 24 (1987) 23–30.

[4] M. Dom, J. Guo, F. Hüffner, R. Niedermeier, Error compensation in leaf root problems, in: Proc. of 15th ISAAC, in: Lecture Notes in Comput. Sci., vol. 3341, Springer, Berlin, 2004, pp. 389–401; Algorithmica, in press.

[5] M. Farber, Characterizations of strongly chordal graphs, Discrete Math. 43 (1983) 173–189.

[6] G.-H. Lin, P.E. Kearney, T. Jiang, Phylogenetic k-root and Steiner k-root, in: Proc. of ISAAC 2000, in: Lecture Notes in Comput. Sci., vol. 1969, Springer, Berlin, 2000, pp. 539–551.

[7] A. Lubiw, Γ-free matrices, Master Thesis, Dept. of Combinatorics and Optimization, University of Waterloo, Canada, 1982.

[8] N. Nishimura, P. Ragde, D.M. Thilikos, On graph powers for leaf-labeled trees, J. Algorithms 42 (2002) 69–108.

[9] D. Rautenbach, Some remarks about leaf roots, Manuscript, 2004.

[10] A. Raychaudhuri, On powers of strongly chordal and circular arc graphs, Ars Combin. 34 (1992) 147–160.

[11] D.J. Rose, R.E. Tarjan, G.S. Lueker, Algorithmic aspects of vertex elimination on graph, SIAM J. Comput. 5 (1976) 266–283.